High-speed cryptography, part 3: more cryptosystems

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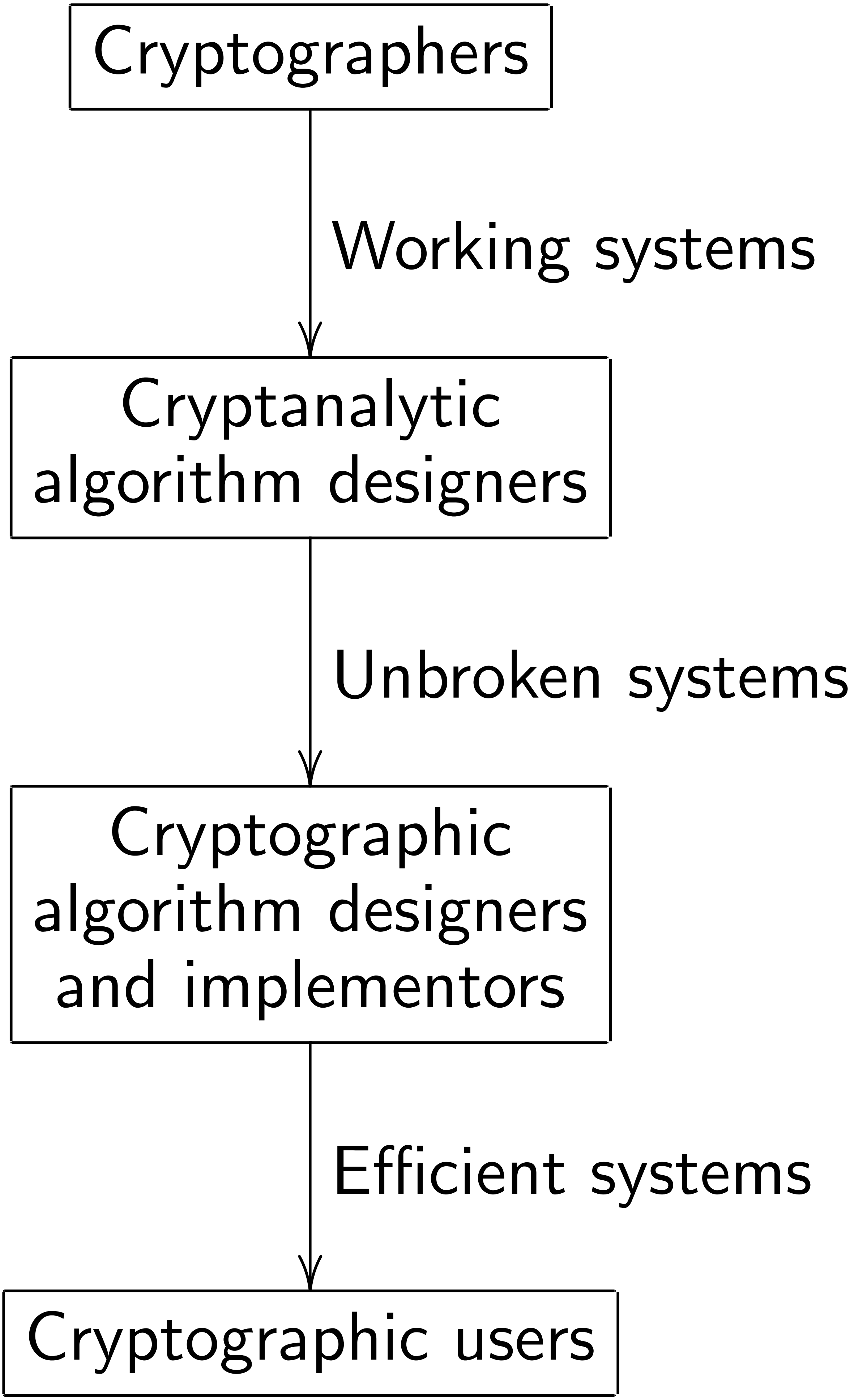
Cryptographers
Working systems
Cryptanalytic algorithm designers
Unbroken systems
Cryptographic algorithm designers and implementors
Efficient systems
Cryptographic users

1. Working systems

Fundamental question for cryptographers:
How can we encrypt, decrypt, sign, verify, etc.?

Many answers:
DES, Triple DES, FEAL-4, AES, RSA, McEliece encryption, Merkle hash-tree signatures, Merkle–Hellman knapsack encryption, Buchmann–Williams class-group encryption, ECDSA, HFE^{v-}, NTRU, et al.

2. Unbroken systems

Fundamental question for
pre-quantum cryptanalysts:
What can an attacker do
using $<2^b$ operations
on a *classical* computer?

Fundamental question for
post-quantum cryptanalysts:
What can an attacker do
using $<2^b$ operations
on a *quantum* computer?

Goal: identify systems that are
not breakable in $<2^b$ operations.

Examples of RSA cryptanalysis:

Schroeppel's "linear sieve", mentioned in 1978 RSA paper, factors pq into p, q using $(2+o(1))^{(\lg pq)^{1/2}(\lg\lg pq)^{1/2}}$ simple operations (conjecturally).

To push this beyond 2^b, must choose pq to have at least $(0.5+o(1))b^2/\lg b$ bits.

Note 1: $\lg = \log_2$.

Note 2: $o(1)$ says *nothing* about, e.g., $b = 128$.
Today: focus on asymptotics.

1993 Buhler–Lenstra–Pomerance, generalizing 1988 Pollard "number-field sieve", factors pq into p, q using $(3.79\ldots + o(1))^{(\lg pq)^{1/3}(\lg \lg pq)^{2/3}}$ simple operations (conjecturally).

To push this beyond 2^b, must choose pq to have at least $(0.015\ldots + o(1))b^3/(\lg b)^2$ bits.

Subsequent improvements: $3.73\ldots$; details of $o(1)$.
But can reasonably conjecture that $2^{(\lg pq)^{1/3+o(1)}}$ is optimal —for classical computers.

Cryptographic systems surviving *pre-quantum* cryptanalysis:

Triple DES (for $b \le 112$), AES-256 (for $b \le 256$), RSA with $b^{3+o(1)}$-bit modulus, McEliece with code length $b^{1+o(1)}$, Merkle signatures with "strong" $b^{1+o(1)}$-bit hash, BW with "strong" $b^{2+o(1)}$-bit discriminant, ECDSA with "strong" $b^{1+o(1)}$-bit curve, HFE$^{\mathsf{v}-}$ with $b^{1+o(1)}$ polynomials, NTRU with $b^{1+o(1)}$ bits, et al.

Typical algorithmic tools for *pre-quantum* cryptanalysts: NFS, ρ, ISD, LLL, F4, XL, et al.

Post-quantum cryptanalysts have all the same tools *plus* quantum algorithms.

Spectacular example:
1994 Shor factors pq into p, q using $(\lg pq)^{2+o(1)}$ simple quantum operations.
To push this beyond 2^b, must choose pq to have at least $2^{(0.5+o(1))b}$ bits. Yikes.

Cryptographic systems surviving *post-quantum* cryptanalysis:

AES-256 (for $b \le 128$),
McEliece code-based encryption with code length $b^{1+o(1)}$,
Merkle hash-based signatures with "strong" $b^{1+o(1)}$-bit hash,
$\mathrm{HFE}^{\mathrm{v}-}$ MQ signatures with $b^{1+o(1)}$ polynomials,
NTRU lattice-based encryption with $b^{1+o(1)}$ bits,
et al.

3. Efficient systems

Fundamental question for designers and implementors of cryptographic algorithms: Exactly how efficient are the unbroken cryptosystems?

Many goals: minimize encryption time, size, decryption time, etc.

Pre-quantum example: RSA encrypts and verifies in $b^{3+o(1)}$ simple operations. Signature occupies $b^{3+o(1)}$ bits.

ECC (with strong curve/$\mathbf{F}_q$, reasonable padding, etc.):

ECDL costs $2^{(1/2+o(1))\lg q}$ by Pollard's rho method. Conjecture: this is the optimal attack against ECC.

Can take $\lg q \in (2+o(1))b$.

Encryption: Fast scalar mult costs $(\lg q)^{2+o(1)} = b^{2+o(1)}$.

Summary: ECC costs $b^{2+o(1)}$. Asymptotically faster than RSA. Bonus: also $b^{2+o(1)}$ *decryption*.

Efficiency is important:
users have cost constraints.

Cryptographers, cryptanalysts,
implementors, etc. tend to
focus on RSA and ECC,
citing these cost constraints.

But Shor breaks RSA and ECC!

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We think that the most efficient unbroken *post-quantum* systems will be hash-based signatures, code-based encryption, lattice-based encryption, multivariate-quadratic sigs.

1978 McEliece system (with length-n classical Goppa codes, reasonable padding, etc.):

Conjecture: Fastest attacks cost $2^{(\beta+o(1))n/\lg n}$.
Quantum attacks: smaller β.

Can take $n \in (1/\beta + o(1))b \lg b$.

Encryption: Matrix mult costs $n^{2+o(1)} = b^{2+o(1)}$.

Summary: McEliece costs $b^{2+o(1)}$.

Hmmm: is this *faster* than ECC? Need more detailed analysis.

ECC encryption:
$\Theta(\lg q)$ operations in $\mathbf{F}_q$.
Each operation in $\mathbf{F}_q$ costs
$\Theta(\lg q \lg \lg q \lg \lg \lg q)$.
Total $\Theta(b^2 \lg b \lg \lg b)$.

ECC encryption:
$\Theta(\lg q)$ operations in $\mathbf{F}_q$.
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McEliece encryption,
with 1986 Niederreiter speedup:
$\Theta(n/\lg n)$ additions in $\mathbf{F}_2^n$,
each costing $\Theta(n)$.
Total $\Theta(b^2 \lg b)$.

ECC encryption:
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Each operation in $\mathbf{F}_q$ costs
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each costing $\Theta(n)$.
Total $\Theta(b^2 \lg b)$.

McEliece is asymptotically faster.
Bonus: Even faster decryption.
Another bonus: Post-quantum.

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1. Speed up ECC: can reduce $\lg \lg b$ using 2007 Fürer; maybe someday eliminate $\lg \lg b$?

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Algorithmic advances can change the competition. Examples:

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3. We're optimizing "subfield AG" variant of McEliece.
Conjecture: Fastest attacks cost $2^{(\alpha+o(1))n}$; encryption $\Theta(b^2)$.

Code-based encryption

Modern version of McEliece:

Receiver's public key is "random" $t \lg n \times n$ matrix K over $\mathbf{F}_2$. Specifies linear $\mathbf{F}_2^n \to \mathbf{F}_2^{t \lg n}$.

Typically $t \lg n \approx 0.2n$; e.g., $n = 2048$, $t = 40$.

Messages suitable for encryption: $\left\{m \in \mathbf{F}_2^n : \#\{i : m_i = 1\} = t\right\}$.

Encryption of m is $Km \in \mathbf{F}_2^{t \lg n}$.

Use hash of m as secret AES-GCM key to encrypt more data.

Attacker, by linear algebra, easily works backwards from Km to *some* $v \in \mathbf{F}_2^n$ such that $Kv = Km$.

i.e. Attacker finds *some* element $v \in m + \text{Ker}K$. Note that $\#\text{Ker}K \geq 2^{n-t\lg n}$.

Attacker wants to decode v: to find element of $\text{Ker}K$ at distance only t from v. Presumably unique, revealing m. But decoding isn't easy!

Receiver builds K with *secret* Goppa structure for fast decoding.

Goppa codes

Fix $q \in \{8, 16, 32, \ldots\}$;
$t \in \{2, 3, \ldots, \lfloor (q-1)/\lg q \rfloor\}$;
$n \in \{t \lg q + 1, t \lg q + 2, \ldots, q\}$.
e.g. $q = 1024$, $t = 50$, $n = 1024$.
or $q = 4096$, $t = 150$, $n = 3600$.

Receiver builds a matrix H as the parity-check matrix for the classical (genus-0) irreducible length-n degree-t binary Goppa code defined by a monic degree-t irreducible polynomial $g \in \mathbf{F}_q[x]$ and distinct $a_1, a_2, \ldots, a_n \in \mathbf{F}_q$.

. . . which means: $H =$

$$\begin{pmatrix} \dfrac{1}{g(a_1)} & \cdots & \dfrac{1}{g(a_n)} \\ \dfrac{a_1}{g(a_1)} & \cdots & \dfrac{a_n}{g(a_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{a_1^{t-1}}{g(a_1)} & \cdots & \dfrac{a_n^{t-1}}{g(a_n)} \end{pmatrix}.$$

View each element of $\mathbf{F}_q$ here as a column in $\mathbf{F}_2^{\lg q}$.

Then $H : \mathbf{F}_2^n \to \mathbf{F}_2^{t \lg q}$.

More useful view: Consider the map $m \mapsto \sum_i m_i/(x - a_i)$ from $\mathbf{F}_2^n$ to $\mathbf{F}_q[x]/g$.

H is the matrix for this map where $\mathbf{F}_2^n$ has standard basis and $\mathbf{F}_q[x]/g$ has basis $\lfloor g/x \rfloor$, $\lfloor g/x^2 \rfloor$, $\ldots$, $\lfloor g/x^t \rfloor$.

One-line proof: In $\mathbf{F}_q[x]$ have

$$\frac{g - g(a_i)}{x - a_i} = \sum_{j \ge 0} a_i^j \left\lfloor g/x^{j+1} \right\rfloor.$$

Receiver generates key K as row reduction of H, revealing only $\mathrm{Ker} H$.

Lattice-based encryption

1998 Hoffstein–Pipher–Silverman NTRU (textbook version, without required padding):

Receiver's public key is "random" $h \in ((\mathbf{Z}/q)[x]/(x^p - 1))^*$.

Ciphertext: $m + rh$ given $m, r \in (\mathbf{Z}/q)[x]/(x^p - 1)$; all coefficients in $\{-1, 0, 1\}$; $\#\{i : r_i{=}{-1}\} = \#\{i : r_i{=}1\} = t$.

p: prime; e.g., $p = 613$.
q: power of 2 around $8p$, with order $\geq (p-1)/2$ in $(\mathbf{Z}/p)^*$.
t: roughly $0.1p$.

Receiver built $h = 3g/(1+3f)$
where $f, g \in (\mathbf{Z}/q)[x]/(x^p - 1)$,
all coeffs in $\{-1, 0, 1\}$,
$\#\{i : f_i{=}{-1}\} = \#\{i : f_i{=}1\} = t$,
$\#\{i : g_i{=}{-1}\} \approx \#\{i : g_i{=}1\} \approx \frac{p}{3}$,
both $1 + 3f$ and g invertible.

Given ciphertext $c = m + rh$,
receiver computes
$(1+3f)c = (1+3f)m + 3rg$
in $(\mathbf{Z}/q)[x]/(x^p - 1)$,
lifts to $\mathbf{Z}[x]/(x^p - 1)$ with
coeffs in $\{-q/2, \ldots, q/2 - 1\}$,
reduces modulo 3
to obtain m.

Basic attack tool:
Lift pairs (u, uh) to $\mathbf{Z}^{2p}$
to obtain a lattice.

Attacking key h:
$(1 + 3f, 3g)$ is a short vector
in this lattice.

Attacking ciphertext c:
$(0, c)$ is close to
lattice vector (r, rh).

Standard lattice algorithms
(SVP, CVP) cost $2^{\Theta(p)}$.
Nothing subexponential known,
even post-quantum.

Take $p \in \Theta(b)$ for security 2^b against all known attacks.

$\Theta(b \lg b)$ bits in key.

Time $b(\lg b)^{2+o(1)}$ to multiply in $(\mathbf{Z}/q)[x]/(x^p - 1)$.

Time $b(\lg b)^{2+o(1)}$ for encryption, decryption.

Excellent overall performance.

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Excellent overall performance.

The McEliece cryptosystem inspires more confidence but has much larger keys.

Something completely different

1985 H. Lange–Ruppert:
$A(\overline{k})$ has a complete system of addition laws, degree $\leq (3, 3)$.
Symmetry $\Rightarrow$ degree $\leq (2, 2)$.

"The proof is nonconstructive. . .
To determine explicitly a complete system of addition laws requires tedious computations already in the easiest case of an elliptic curve in Weierstrass normal form."

1985 Lange–Ruppert:
Explicit complete system
of 3 addition laws
for short Weierstrass curves.

Reduce formulas to 53 monomials
by introducing extra variables
$x_i y_j + x_j y_i$, $x_i y_j - x_j y_i$.

1987 Lange–Ruppert:
Explicit complete system
of 3 addition laws
for long Weierstrass curves.

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3) X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4) X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1) X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2) X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6) X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1) Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6) Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6) X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6) X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2) Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2) Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1) Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1) Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2 + (a_3^3 + 3a_3 a_6) Z_1^2 Z_2^2.
\end{aligned}$$

1995 Bosma–Lenstra:
Explicit complete system
of 2 addition laws
for long Weierstrass curves:

$$X_3, Y_3, Z_3, X'_3, Y'_3, Z'_3$$
$$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$$
$$X_1, Y_1, Z_1, X_2, Y_2, Z_2].$$

1995 Bosma–Lenstra:
Explicit complete system
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for long Weierstrass curves:

$$X_3, Y_3, Z_3, X_3', Y_3', Z_3'$$
$$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$$
$$X_1, Y_1, Z_1, X_2, Y_2, Z_2].$$

My previous slide in this talk:
Bosma–Lenstra Y_3', Z_3'.

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$X_3, Y_3, Z_3, X_3', Y_3', Z_3'$
$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$
$\quad X_1, Y_1, Z_1, X_2, Y_2, Z_2]$.

My previous slide in this talk:
Bosma–Lenstra Y_3', Z_3'.
Actually, slide shows
$\text{Publish}(Y_3'), \text{Publish}(Z_3')$,
where Publish introduces typos.

What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass curves
$E/k : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$,
all $P_1 = (X_1 : Y_1 : Z_1) \in E(k)$,
all $P_2 = (X_2 : Y_2 : Z_2) \in E(k)$:

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
$(X_3' : Y_3' : Z_3')$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
at most one of these is $(0 : 0 : 0)$.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwards curves E/k :
$X^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2$,
all $P_1, P_2 \in E(k)$,
$P_1 = ((X_1 : Z_1), (Y_1 : T_1))$,
$P_2 = ((X_2 : Z_2), (Y_2 : T_2))$:

$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(X_3' : Z_3')$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3' : T_3')$ is $y(P_1 + P_2)$ or $(0 : 0)$;
at most one of these is $(0 : 0)$.

$$X_3 = X_1Y_2Z_2T_1 + X_2Y_1Z_1T_2,$$
$$Z_3 = Z_1Z_2T_1T_2 + dX_1X_2Y_1Y_2,$$
$$Y_3 = Y_1Y_2Z_1Z_2 - X_1X_2T_1T_2,$$
$$T_3 = Z_1Z_2T_1T_2 - dX_1X_2Y_1Y_2,$$

$$X_3' = X_1Y_1Z_2T_2 + X_2Y_2Z_1T_1,$$
$$Z_3' = X_1X_2T_1T_2 + Y_1Y_2Z_1Z_2,$$
$$Y_3' = X_1Y_1Z_2T_2 - X_2Y_2Z_1T_1,$$
$$T_3' = X_1Y_2Z_2T_1 - X_2Y_1Z_1T_2.$$

Much, much, much simpler than Lange–Ruppert, Bosma–Lenstra.

Also much easier to prove.

5. Explicit Formulae

From [5, Chapter III, 2.3] it follows that $f = m^*(X/Z)$ and $g = m^*(Y/Z)$ are given by

$$f = \lambda^2 + a_1\lambda - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2, \qquad g = -(\lambda + a_1) f - \nu - a_3,$$

where

$$\lambda = \frac{Y_1Z_2 - Y_2Z_1}{X_1Z_2 - X_2Z_1} \qquad \text{and} \qquad \nu = -\frac{Y_1X_2 - Y_2X_1}{X_1Z_2 - X_2Z_1}.$$

Applying the automorphism of $E \times E$ mapping (P_1, P_2) to $(P_1, -P_2)$ we find that

$$s^*(X/Z) = \kappa^2 + a_1\kappa - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2$$

and

$$s^*(Y/Z) = -(\kappa + a_1)\, s^*(X/Z) - \mu - a_3,$$

where

$$\kappa = \frac{Y_1Z_2 + Y_2Z_1 + a_1X_2Z_1 + a_3Z_1Z_2}{X_1Z_2 - X_2Z_1}$$

and

$$\mu = -\frac{Y_1X_2 + Y_2X_1 + a_1X_1X_2 + a_3X_1Z_2}{X_1Z_2 - X_2Z_1}.$$

The bijection of Theorem 2 maps $(0:0:1)$ to the addition law given by $X_3^{(1)} = fZ_0$, $Y_3^{(1)} = gZ_0$, $Z_3^{(1)} = Z_0$, which in explicit terms is found to be given by

$$\begin{aligned}
X_3^{(1)} = {} & (X_1Y_2 - X_2Y_1)(Y_1Z_2 + Y_2Z_1) + (X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\
& + a_1X_1X_2(Y_1Z_2 - Y_2Z_1) + a_1(X_1Y_2 - X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& - a_2X_1X_2(X_1Z_2 - X_2Z_1) + a_3(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\
& + a_3(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& - a_4(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& - 3a_6(X_1Z_2 - X_2Z_1)\, Z_1Z_2,
\end{aligned}$$

$$\begin{aligned}
Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2 - X_2Y_1) \\
& - Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(X_1Z_2 - X_2Z_1)\,Y_1Y_2 \\
& + (a_1^2 + 3a_2)\,X_1X_2(Y_1Z_2 - Y_2Z_1) \\
& - (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1a_2 - 3a_3)\,X_1X_2(X_1Z_2 - X_2Z_1) \\
& - (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\,Z_1Z_2 \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z_1)\,Z_1Z_2, \\
Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + a_1(X_1Y_2 - X_2Y_1)\,Z_1Z_2 - a_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 - X_2Z_1)\,Z_1Z_2.
\end{aligned}$$

The corresponding exceptional divisor is $3 \cdot \Delta$, so a pair of points P_1, P_2 on E is exceptional for this addition law if and only if $P_1 = P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we obtain the addition law corresponding to $(0:1:0)$. It reads as follows:

$$\begin{aligned}
X_3^{(2)} = {} & Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 + X_2Y_1)\,X_2Y_1 + a_1^2X_1X_2^2Y_1 \\
& - a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + 2Y_2Z_1) \\
& + a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& - a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& - a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1Z_2 + X_2Z_1) \\
& - a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z_1 + Y_1Z_2) \\
& - 3a_6(X_1Y_2 + X_2Y_1)\,Z_1Z_2 \\
& - 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_3a_4(X_1Z_2 - 2X_2Z_1)\,X_2Z_1 \\
& - (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2)(Y_1Z_2 + Y_2Z_1)\,Z_1Z_2 \\
& - (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1a_2a_6 - a_1a_4^2)\,X_1Z_1Z_2^2 \\
& - a_3^3(X_1Z_2 + X_2Z_1)\,Z_1Z_2 - 3a_3a_6(X_1Z_2 + 2X_2Z_1)\,Z_1Z_2 \\
& - (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2a_3a_6 - a_3a_4^2)\,Z_1^2Z_2^2,
\end{aligned}$$

$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3) X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4) X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1) X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2) X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6) X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1) Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6) Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6) X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6) X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2) Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2) Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1) Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1) Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2 + (a_3^3 + 3a_3 a_6) Z_1^2 Z_2^2.
\end{aligned}
$$

1987 Lenstra: Use Lange–Ruppert complete system of addition laws to computationally define group $E(R)$ for more general rings R—rings with trivial class group.

Define $\mathbf{P}^2(R) = \{(X : Y : Z) : X, Y, Z \in R;\ XR + YR + ZR = R\}$ where $(X : Y : Z)$ is the module $\{(\lambda X, \lambda Y, \lambda Z) : \lambda \in R\}$.

Define $E(R) = \{(X : Y : Z) \in \mathbf{P}^2(R) : Y^2 Z = X^3 + a_4 X Z^2 + a_6 Z^3\}$.

To define (and compute) sum
$(X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z_2)$:

Consider (and compute)
Lange–Ruppert $(X_3 : Y_3 : Z_3)$,
$(X_3' : Y_3' : Z_3')$, $(X_3'' : Y_3'' : Z_3'')$.

Add these R-modules:

$$\{ \quad (\lambda X_3, \lambda Y_3, \lambda Z_3) + (\lambda' X_3', \lambda' Y_3', \lambda' Z_3') + (\lambda'' X_3'', \lambda'' Y_3'', \lambda'' Z_3'') : \lambda, \lambda', \lambda'' \in R\}.$$

Express as $(X : Y : Z)$,
using trivial class group of R.