Twisted Hessian curves

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University of Illinois at Chicago &
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Joint work with:

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David Kohel
Aix-Marseille Université

Tanja Lange
Technische Universiteit Eindhoven

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"The crucial problem becomes the choice of the model of an algebraic group variety, where computations mod p are the least time consuming."

Most important computations:
ADD is $P, Q \mapsto P + Q$.
DBL is $P \mapsto 2P$.

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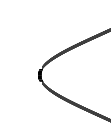

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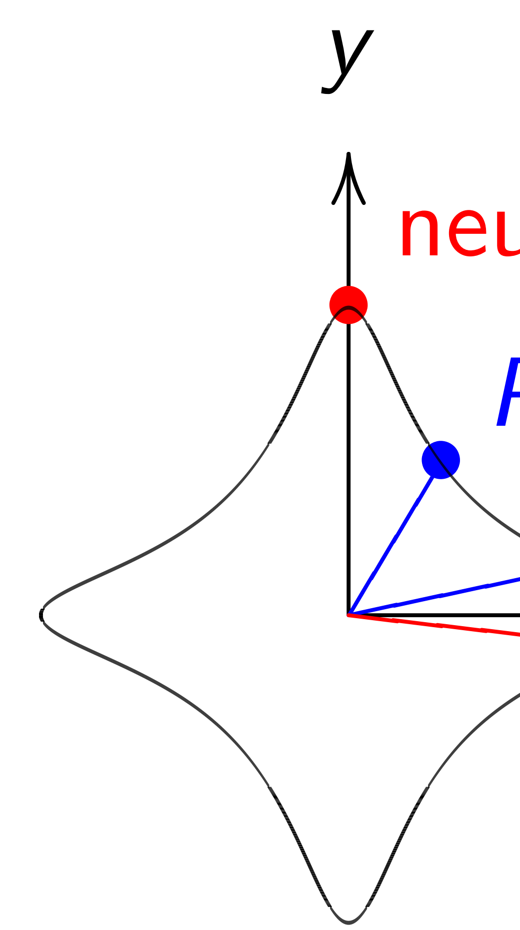

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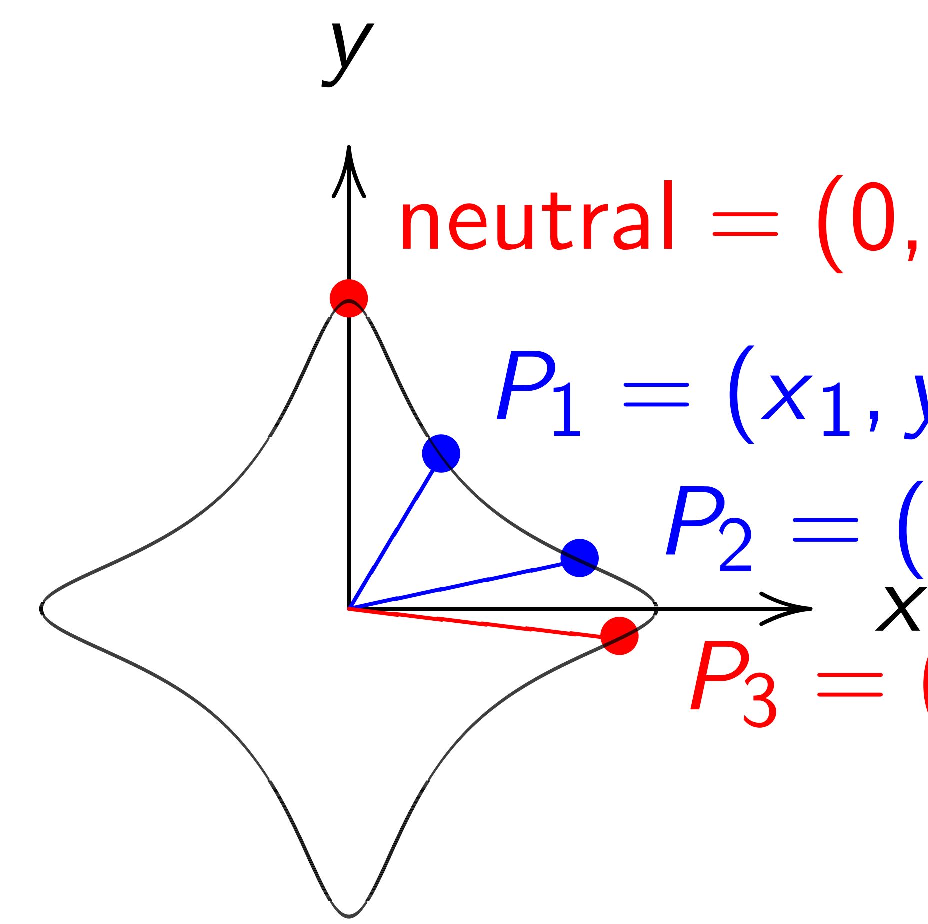

Example: $x^2 + y^2 = 1 - 30$

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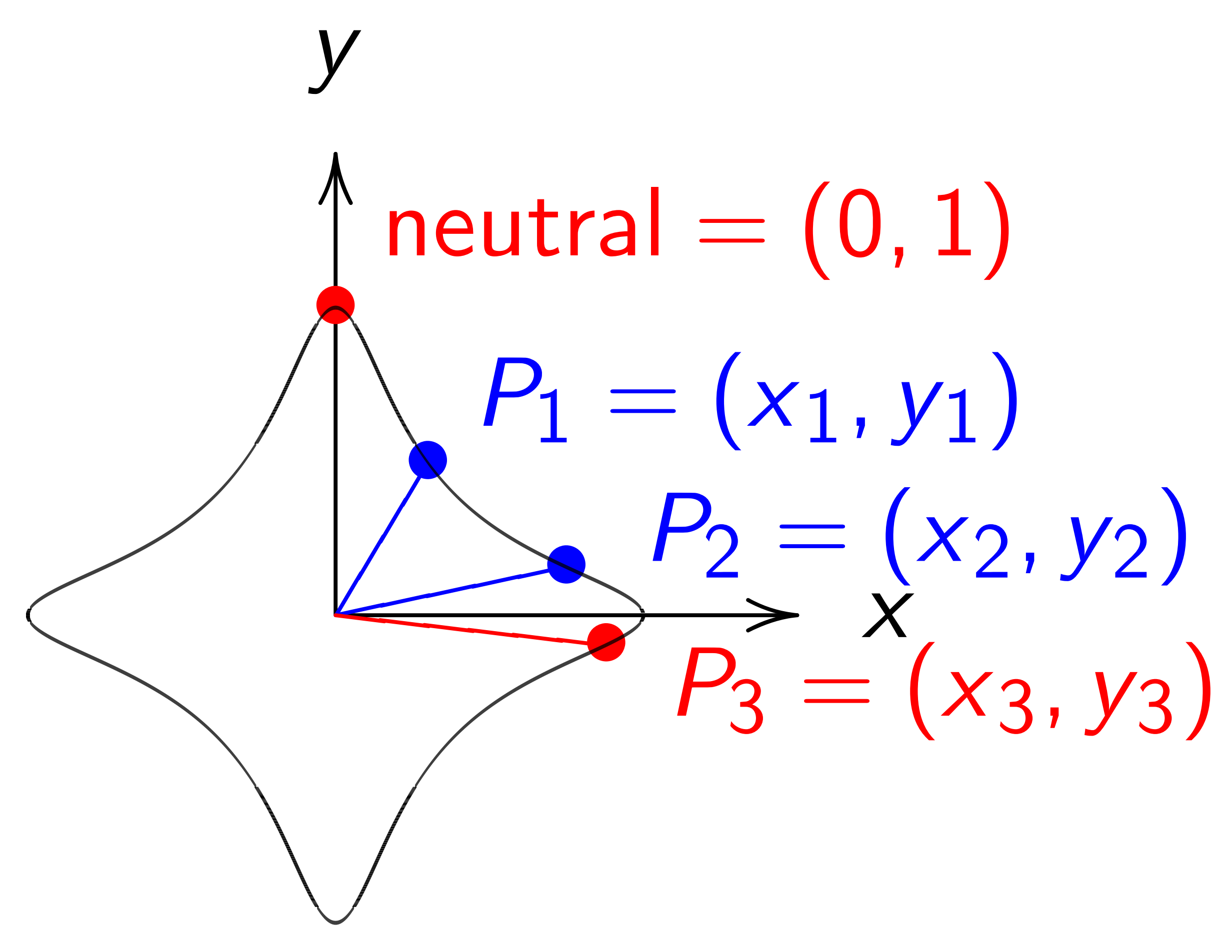

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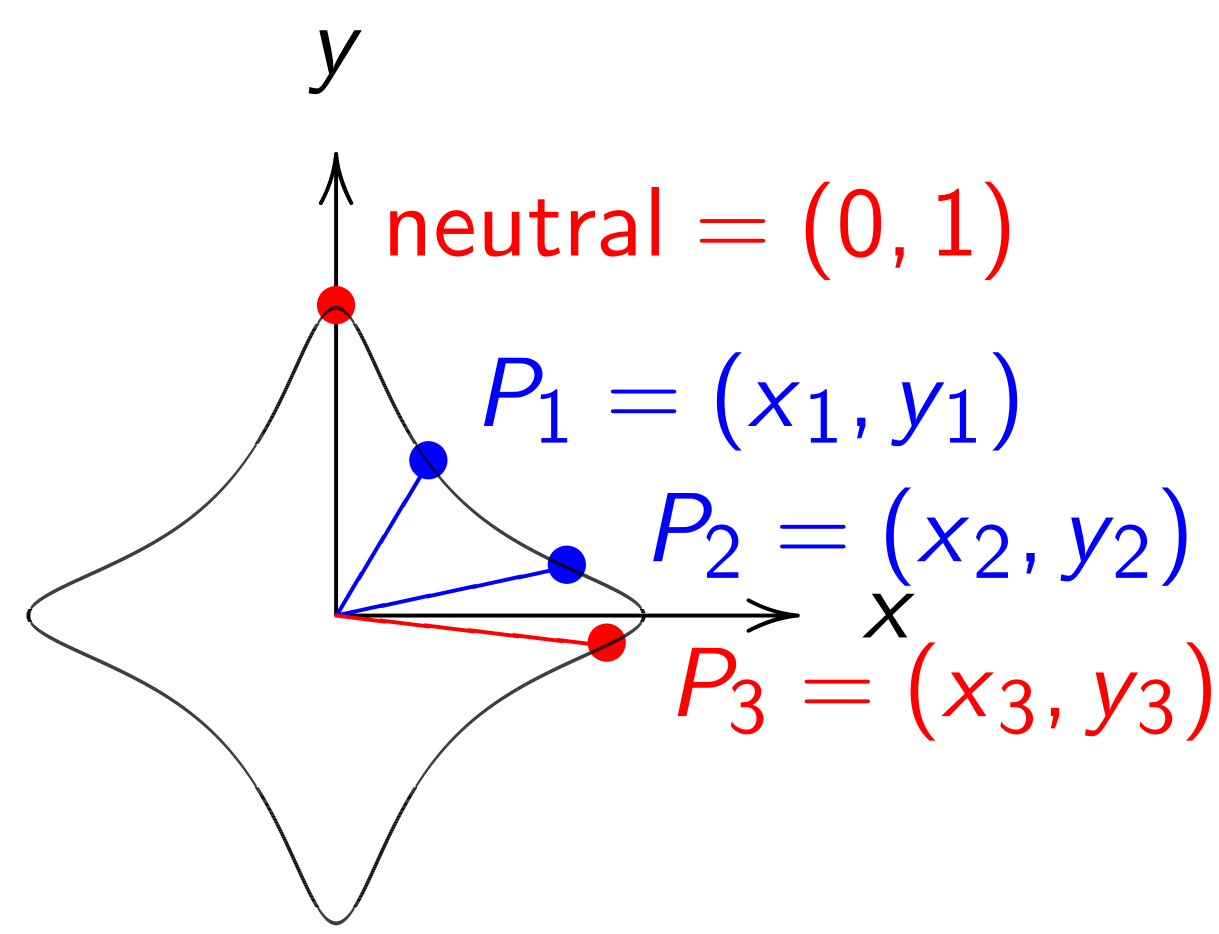

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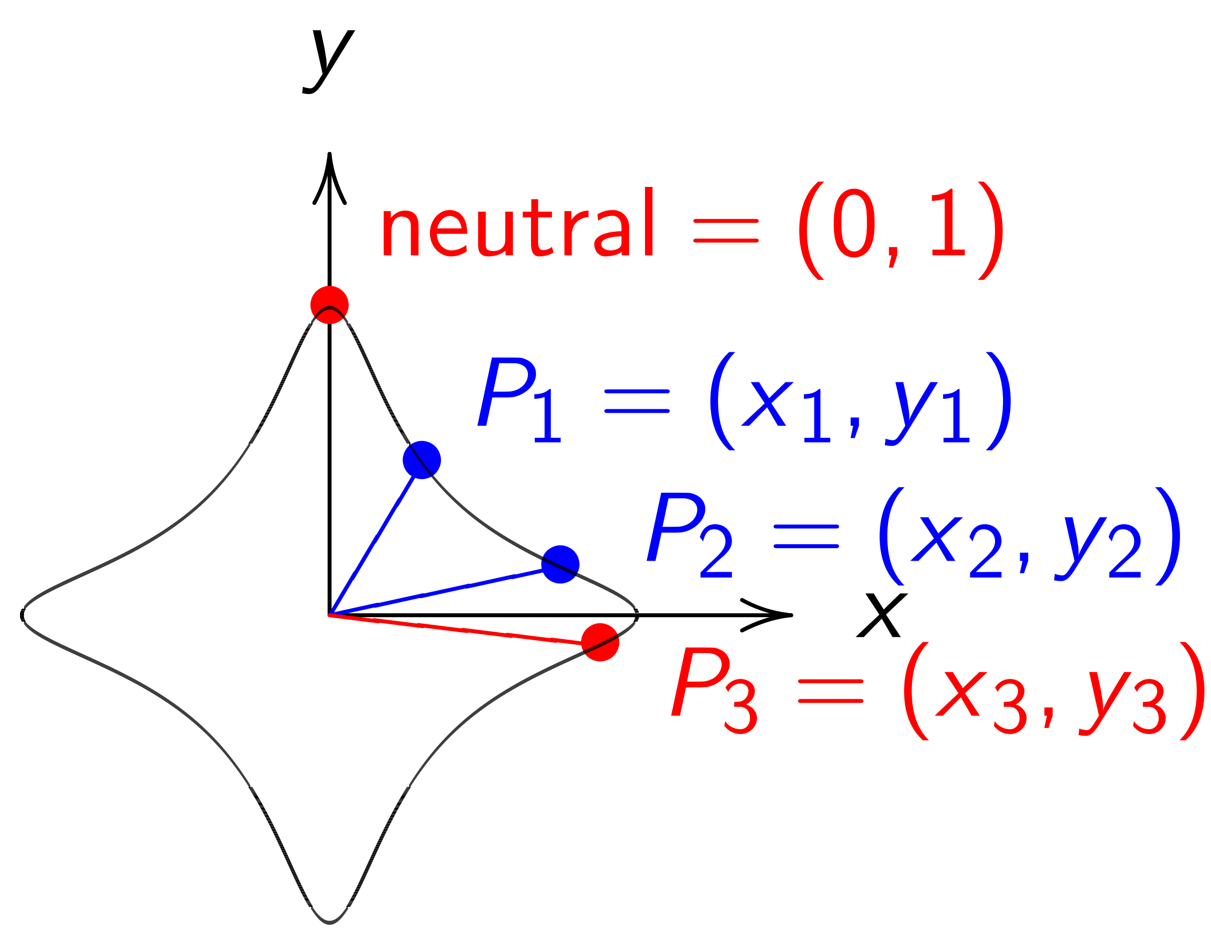

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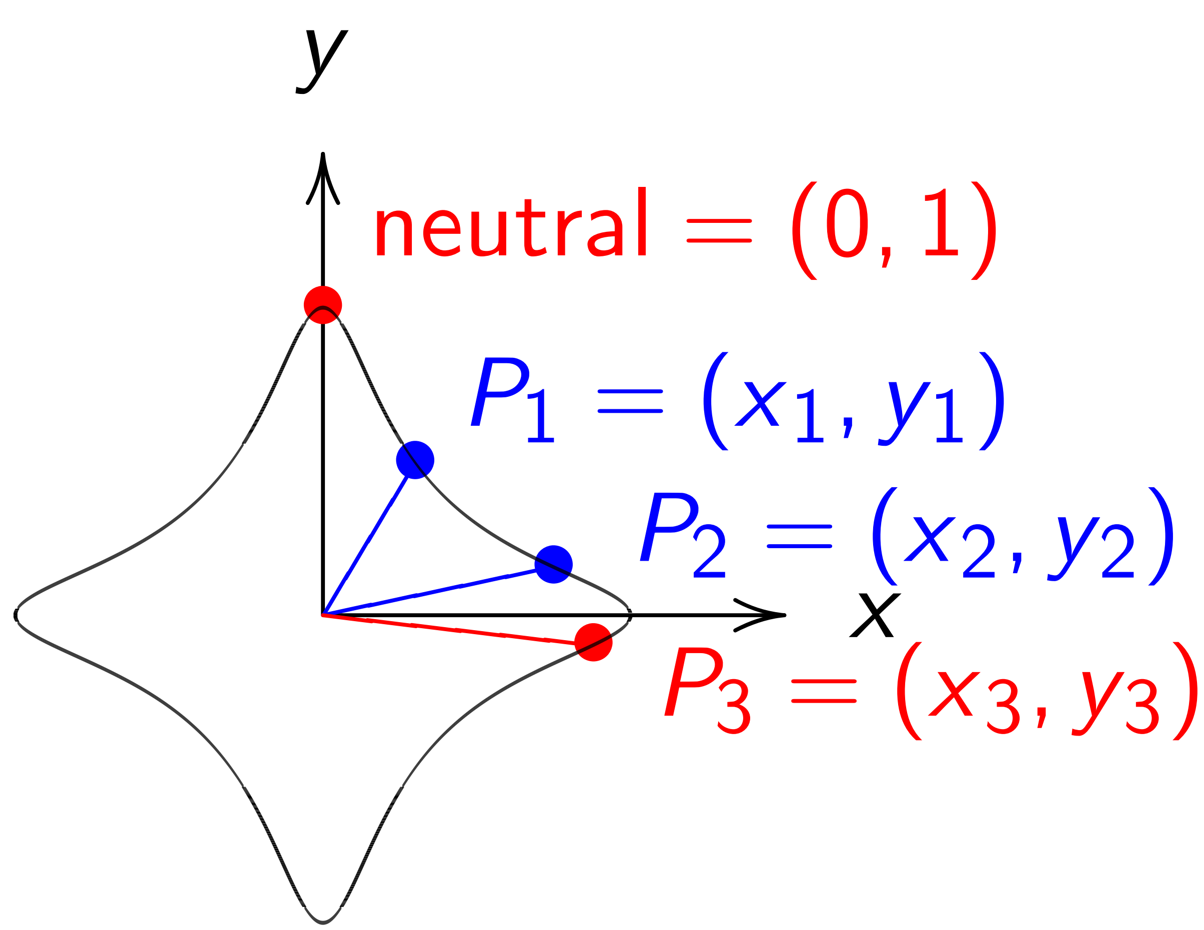

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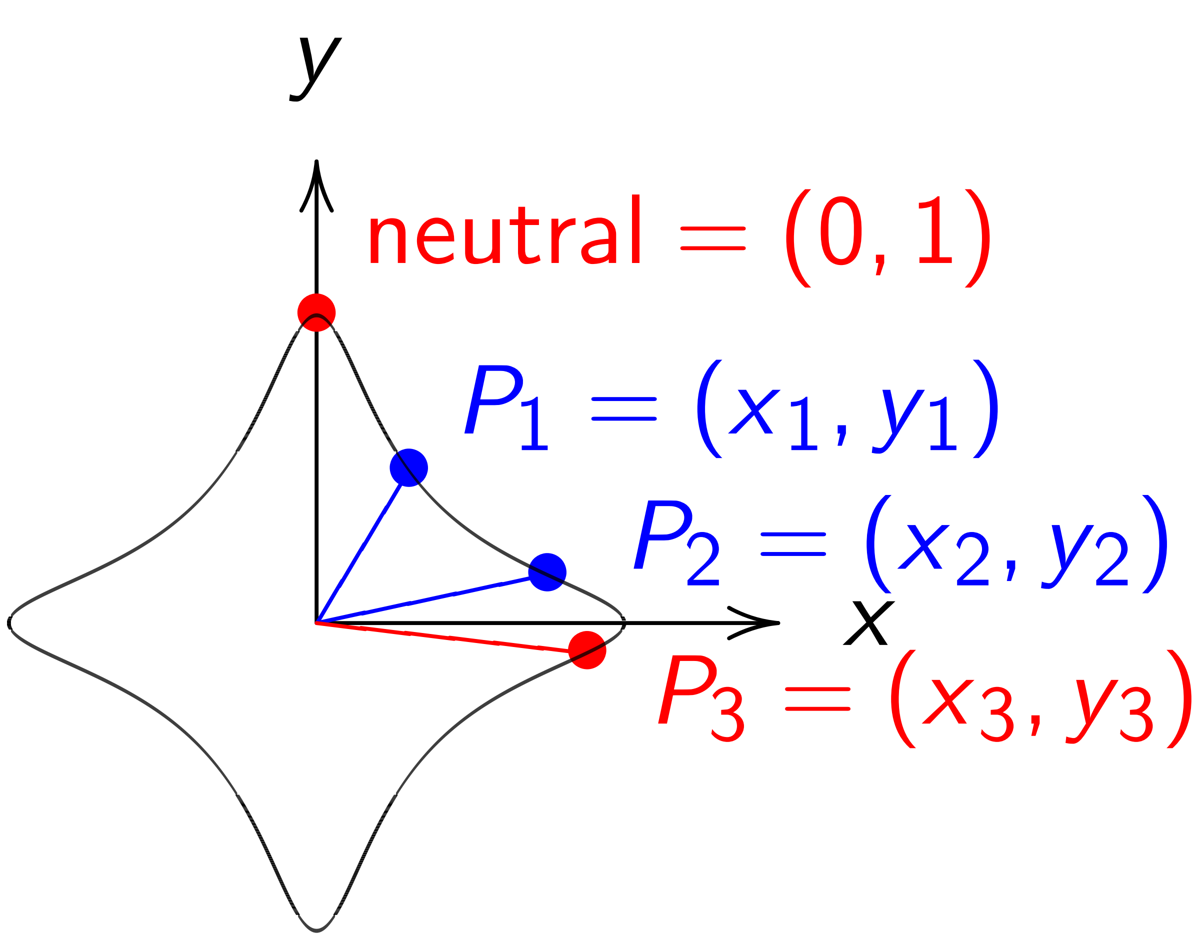

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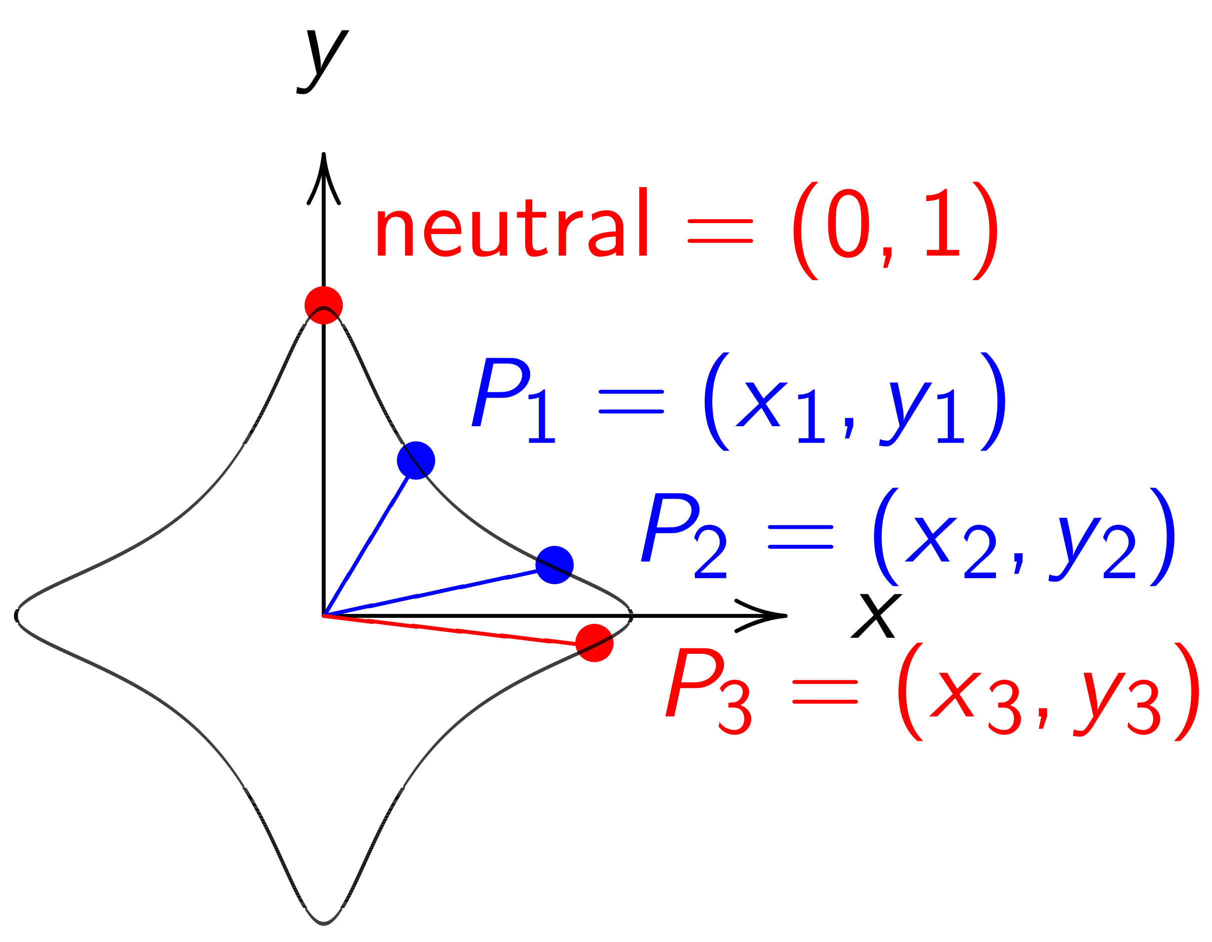

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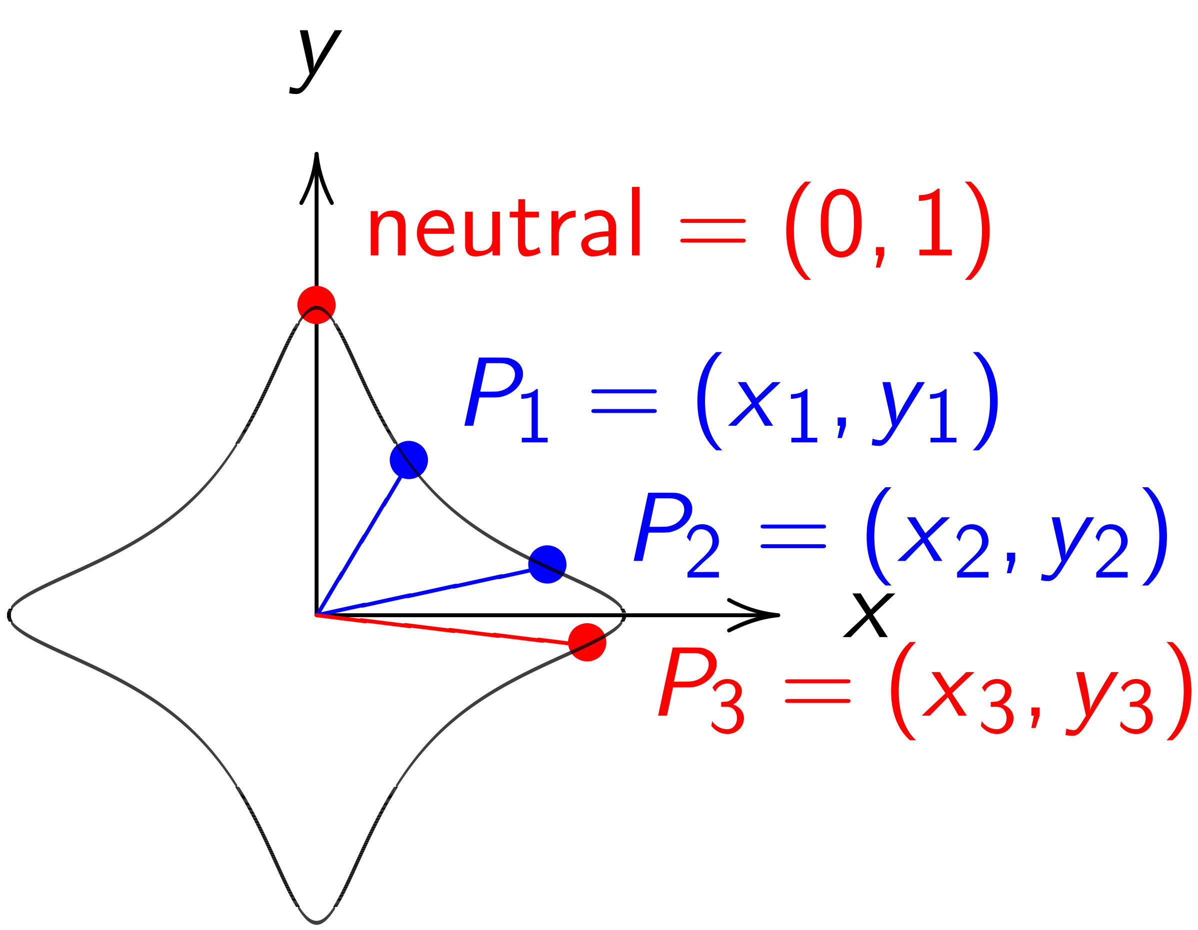

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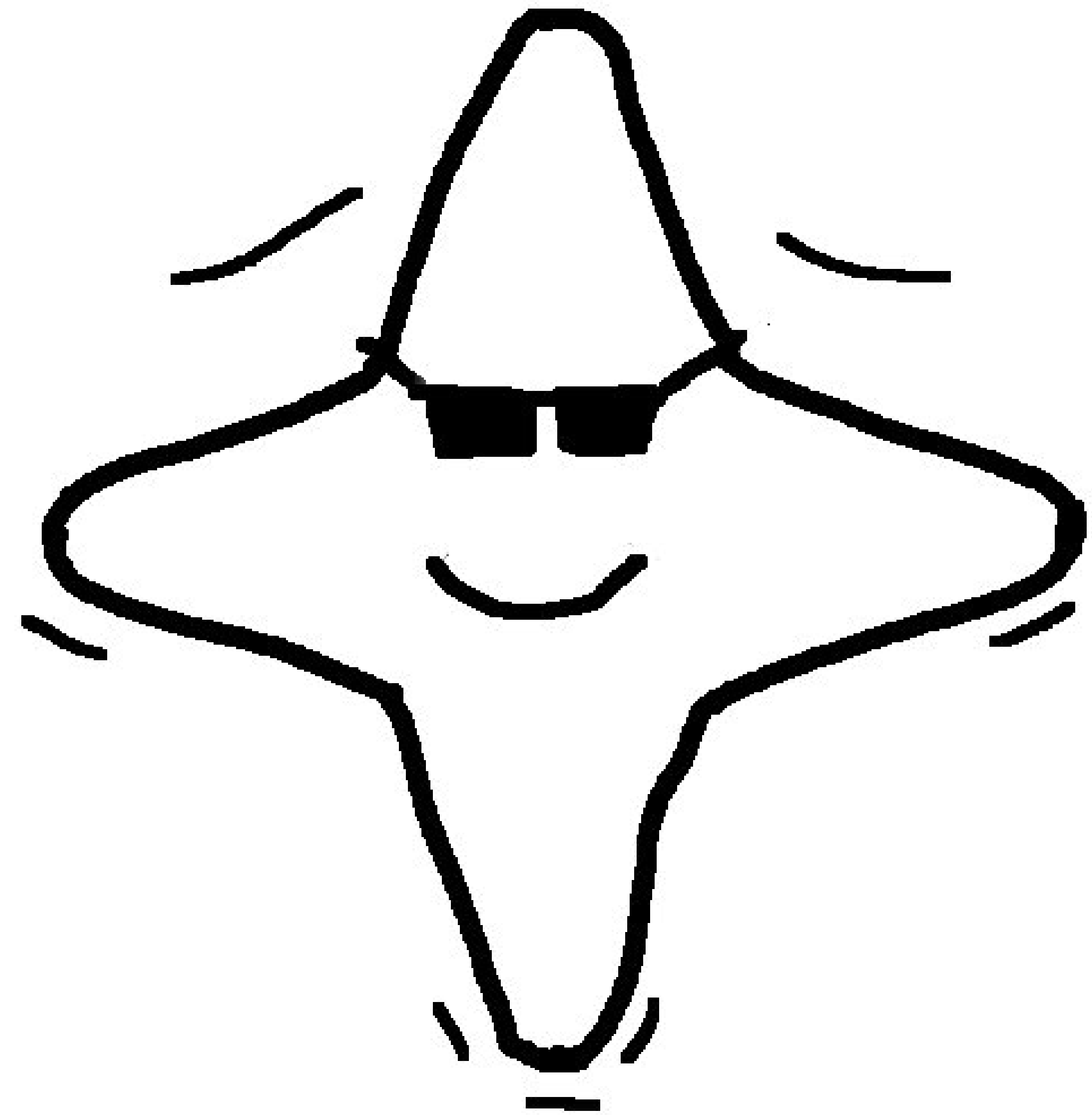
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

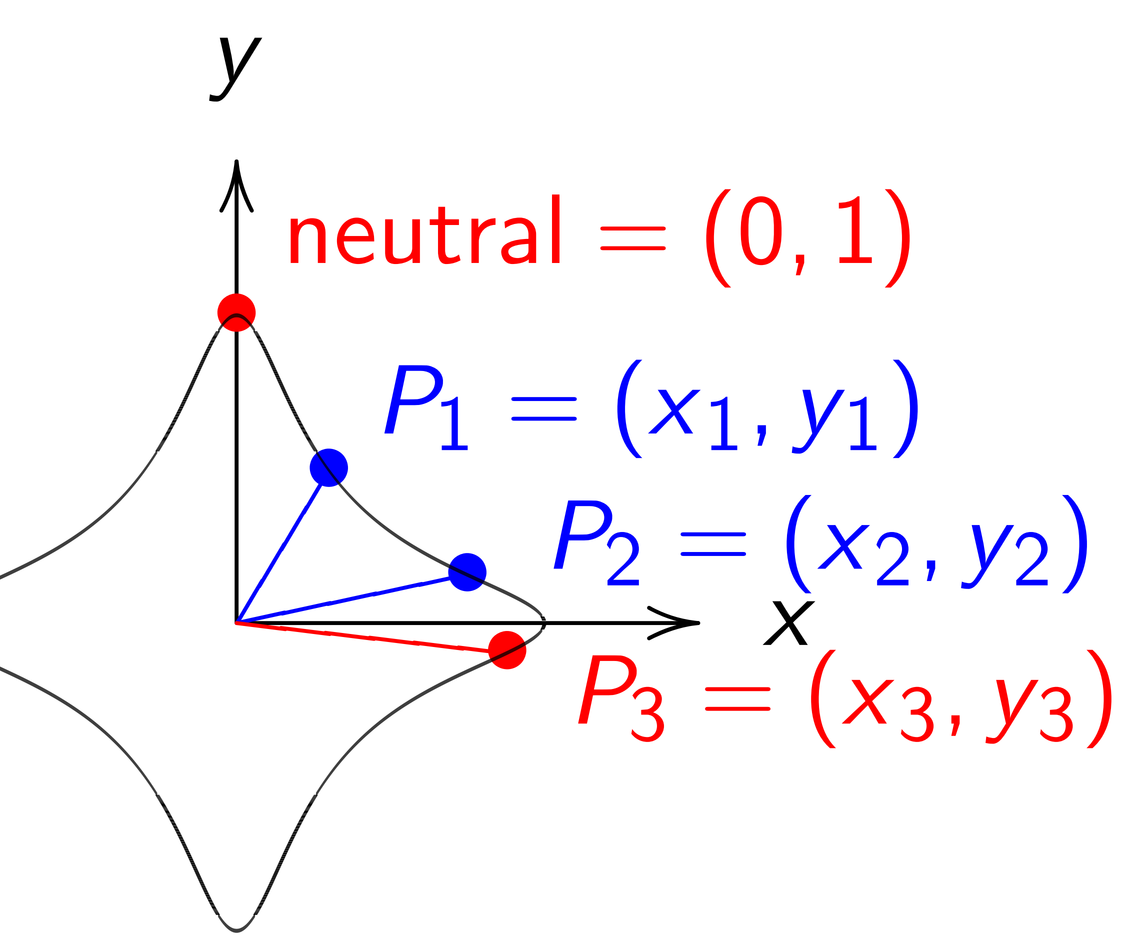
2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.



wards: new curve shape.

rnstein–Lange: generalize,

speed, completeness.



e: $x^2 + y^2 = 1 - 30x^2y^2$.

(x_1, y_1) and (x_2, y_2) is

$+y_1x_2)/(1-30x_1x_2y_1y_2)$,

$x_1x_2)/(1+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

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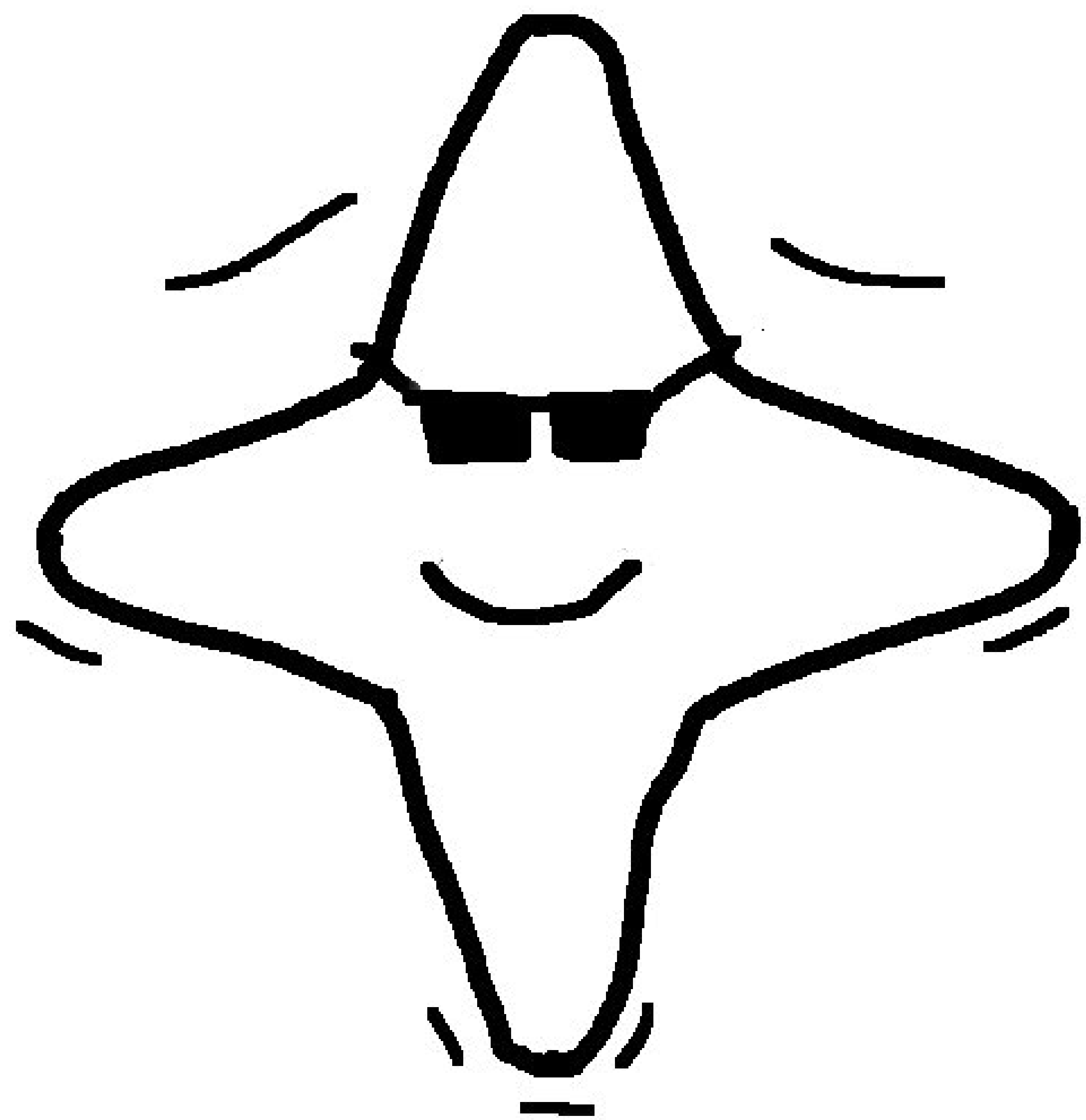
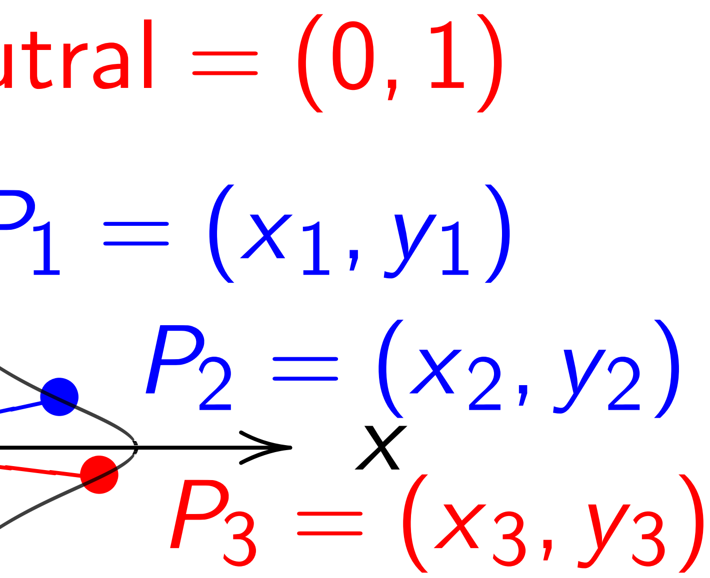


$y^2 = x^3$

w curve shape.
ange: generalize,
mpleteness.



$= 1 - 30x^2y^2$.
nd (x_2, y_2) is
$-30x_1x_2y_1y_2)$,
$+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

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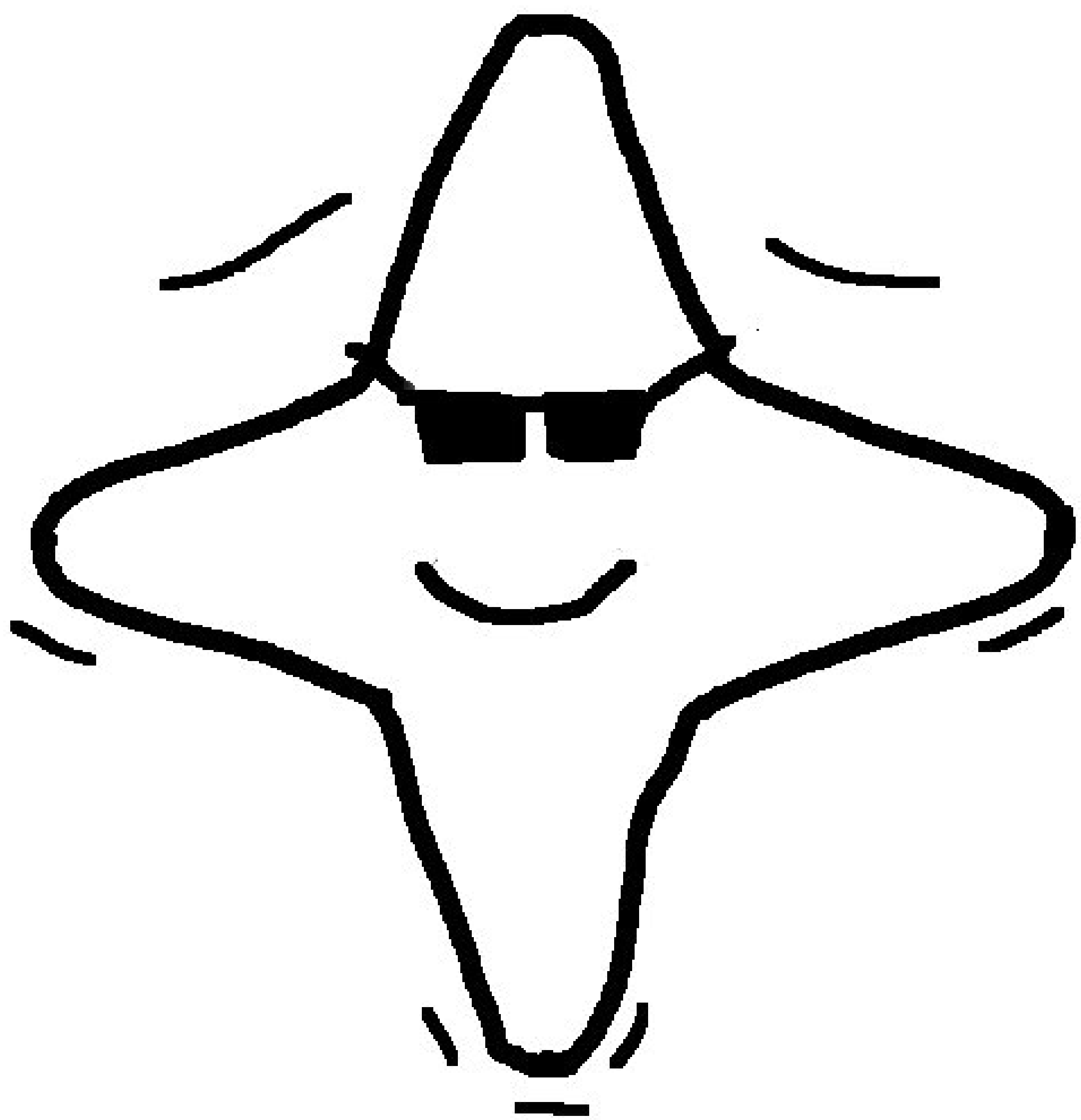
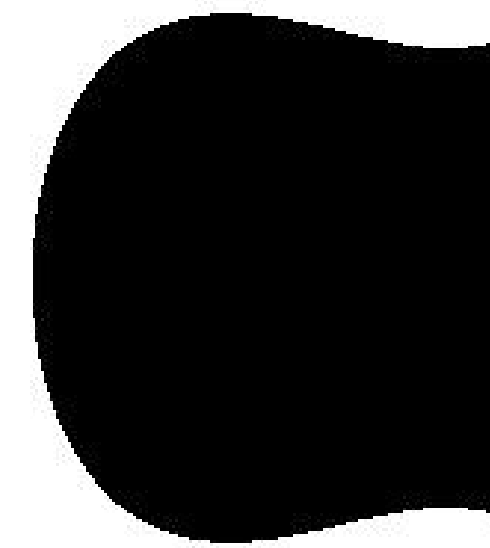
just 8**M** for ADD.





$y^2 = x^3 - 0.4x +$

hape.

eralize,

s.

$1)$

$y_1)$

$x_2, y_2)$

(x_3, y_3)

$x^2 y^2$.

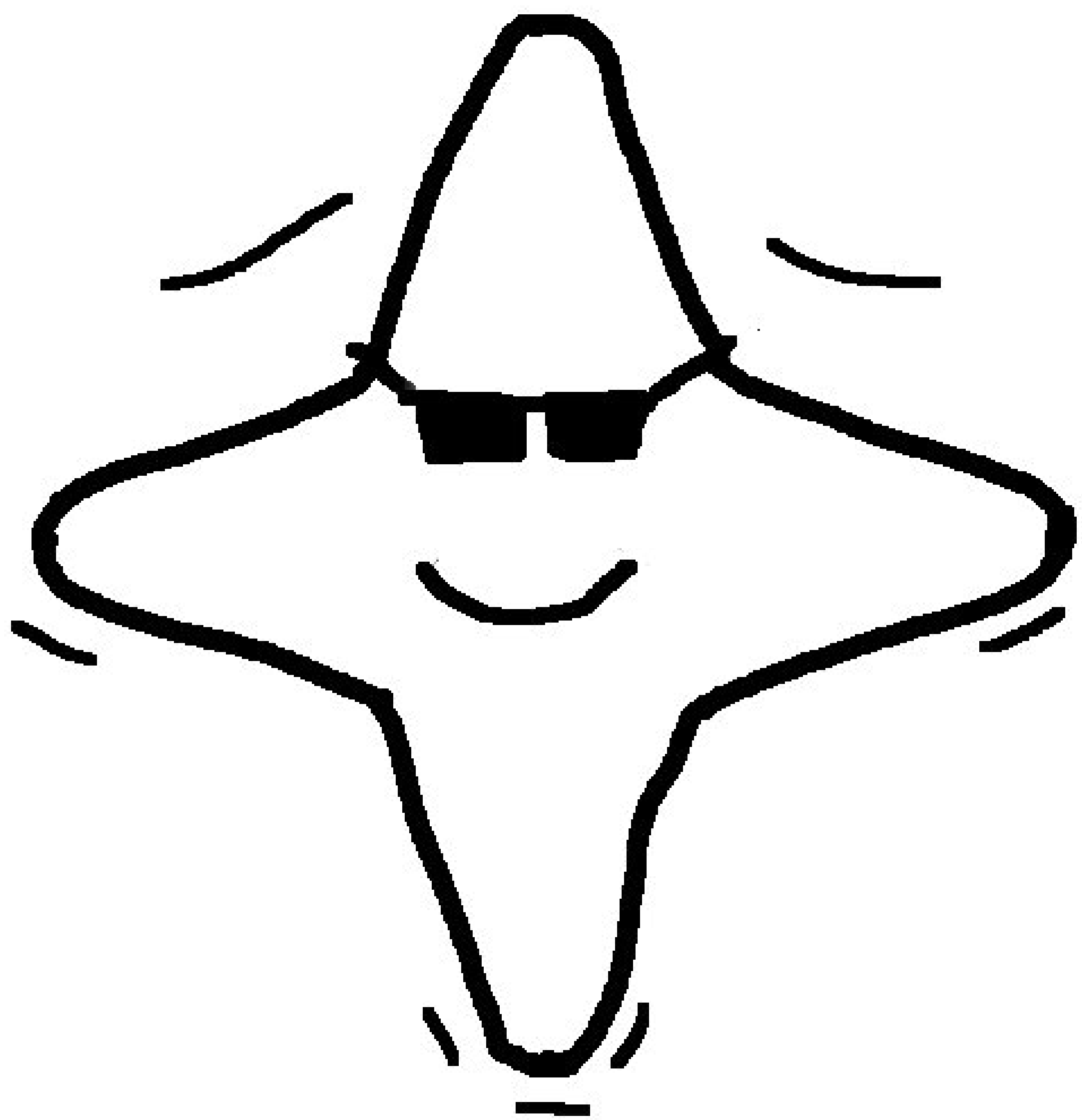
$)$ is

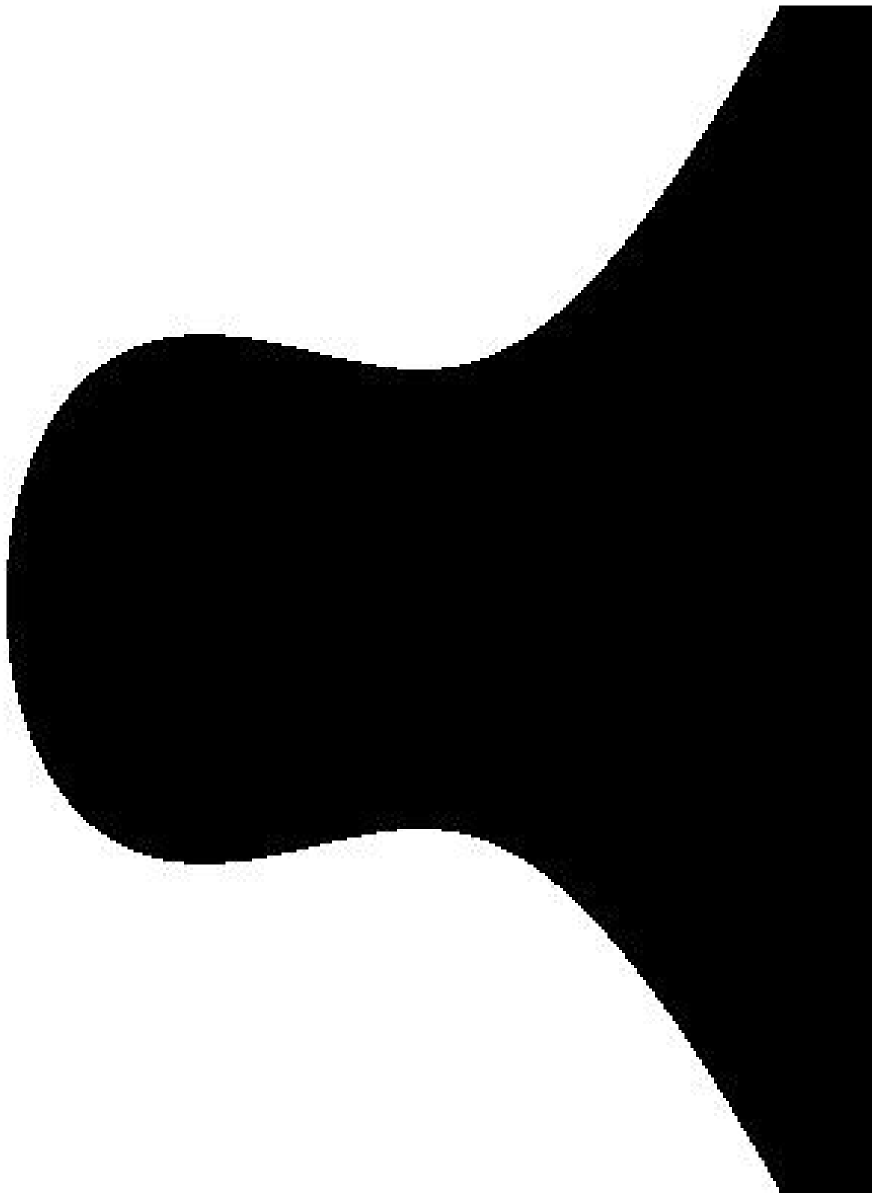
$_1 y_2)$,

$_1 y_2))$.

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.

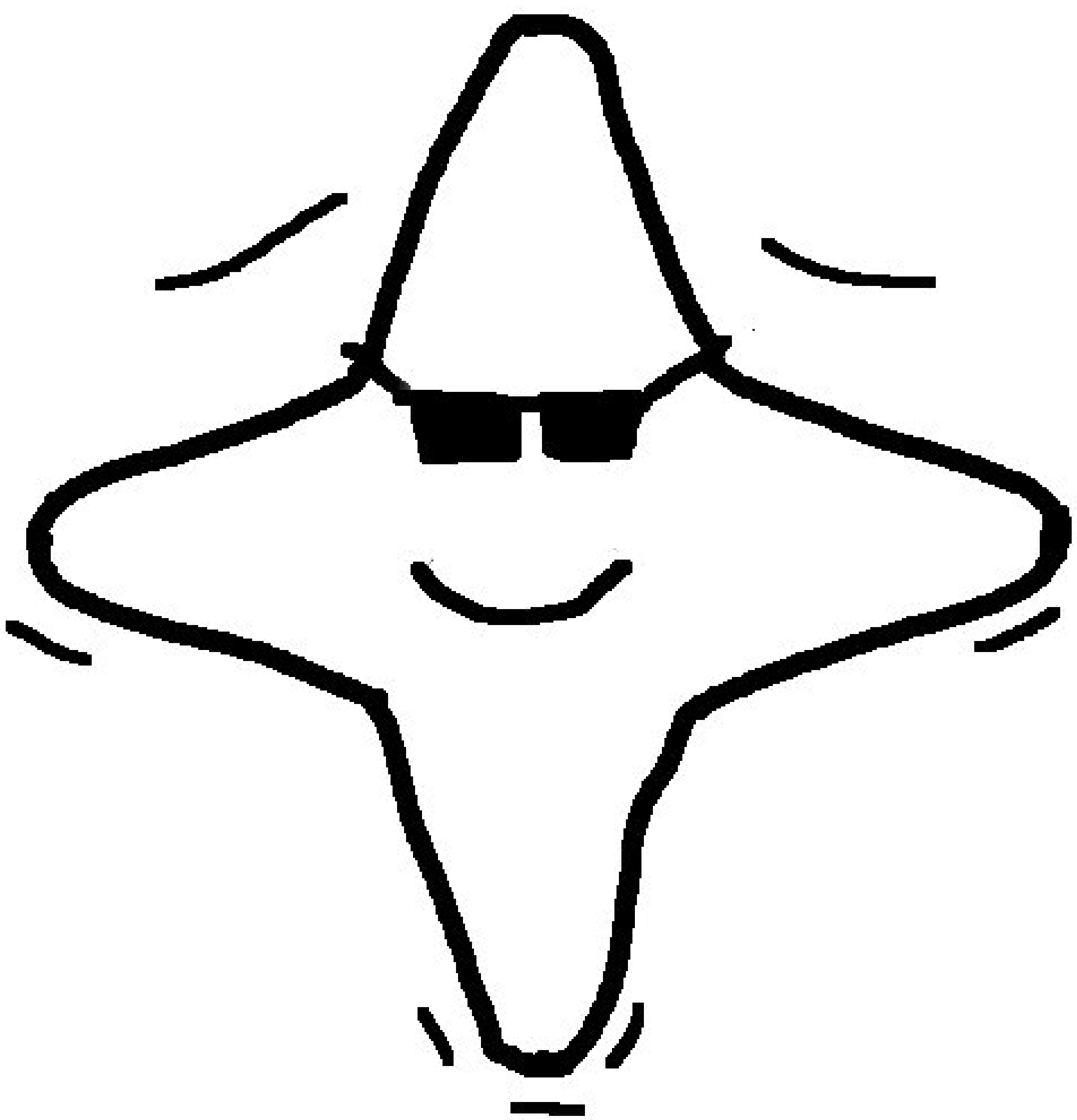


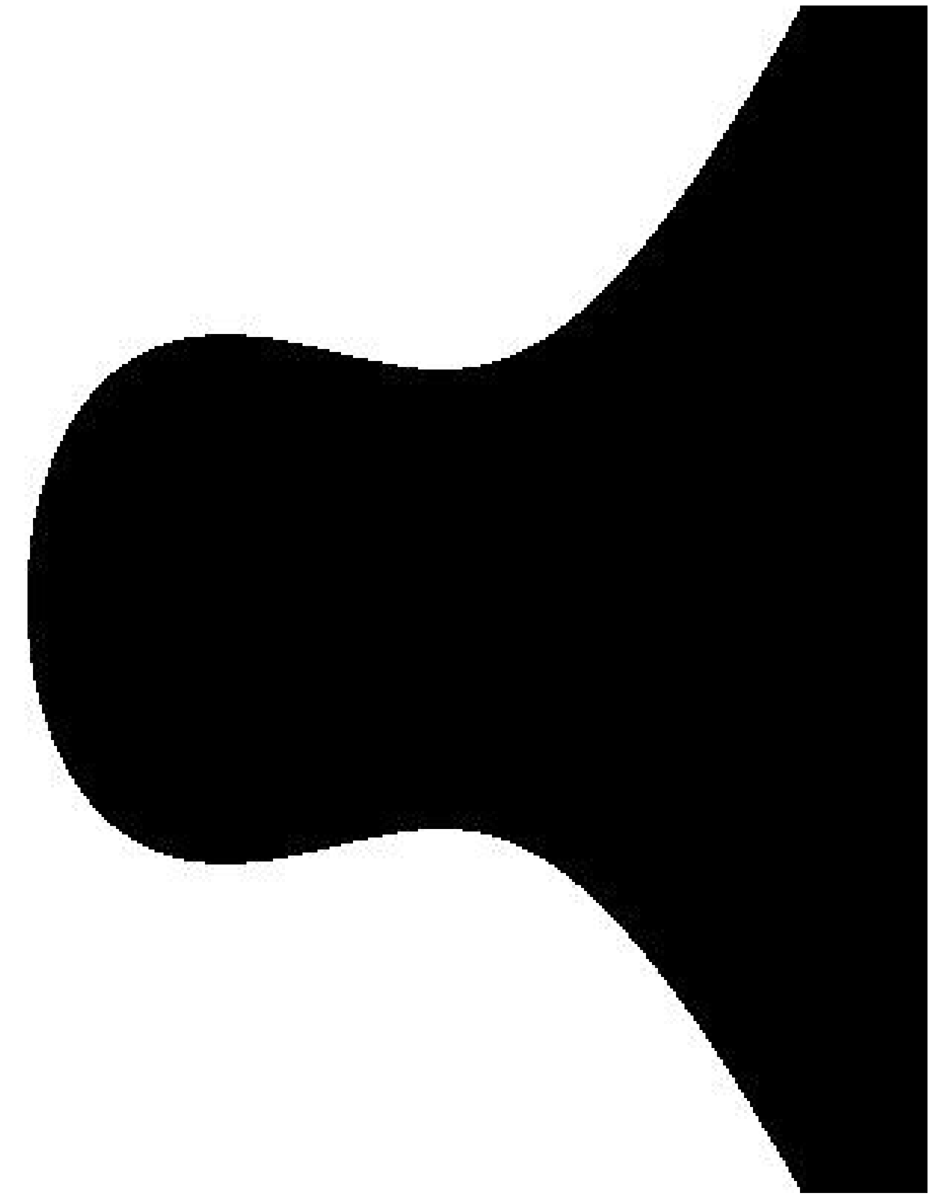


$$y^2 = x^3 - 0.4x + 0.7$$

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.

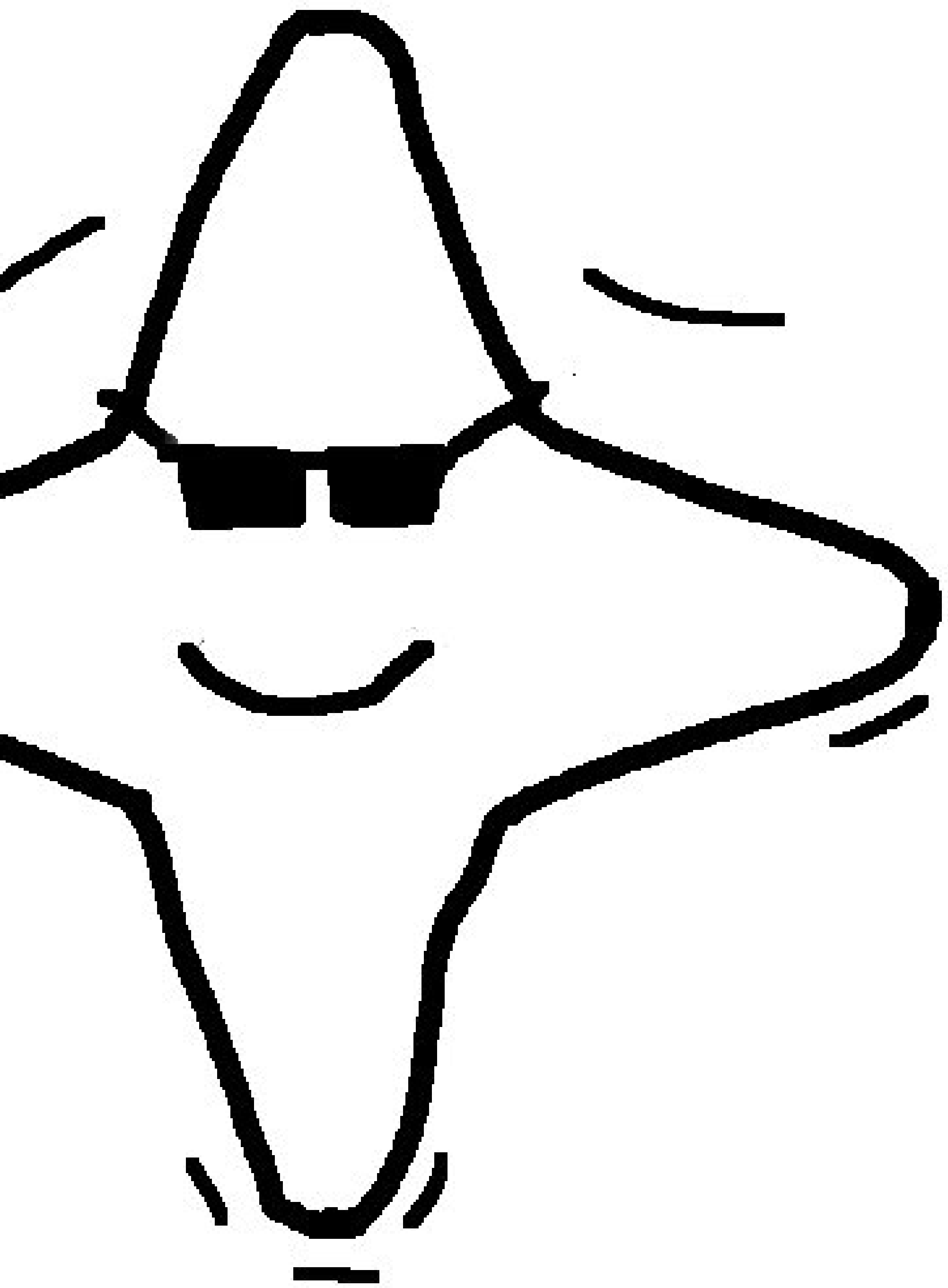


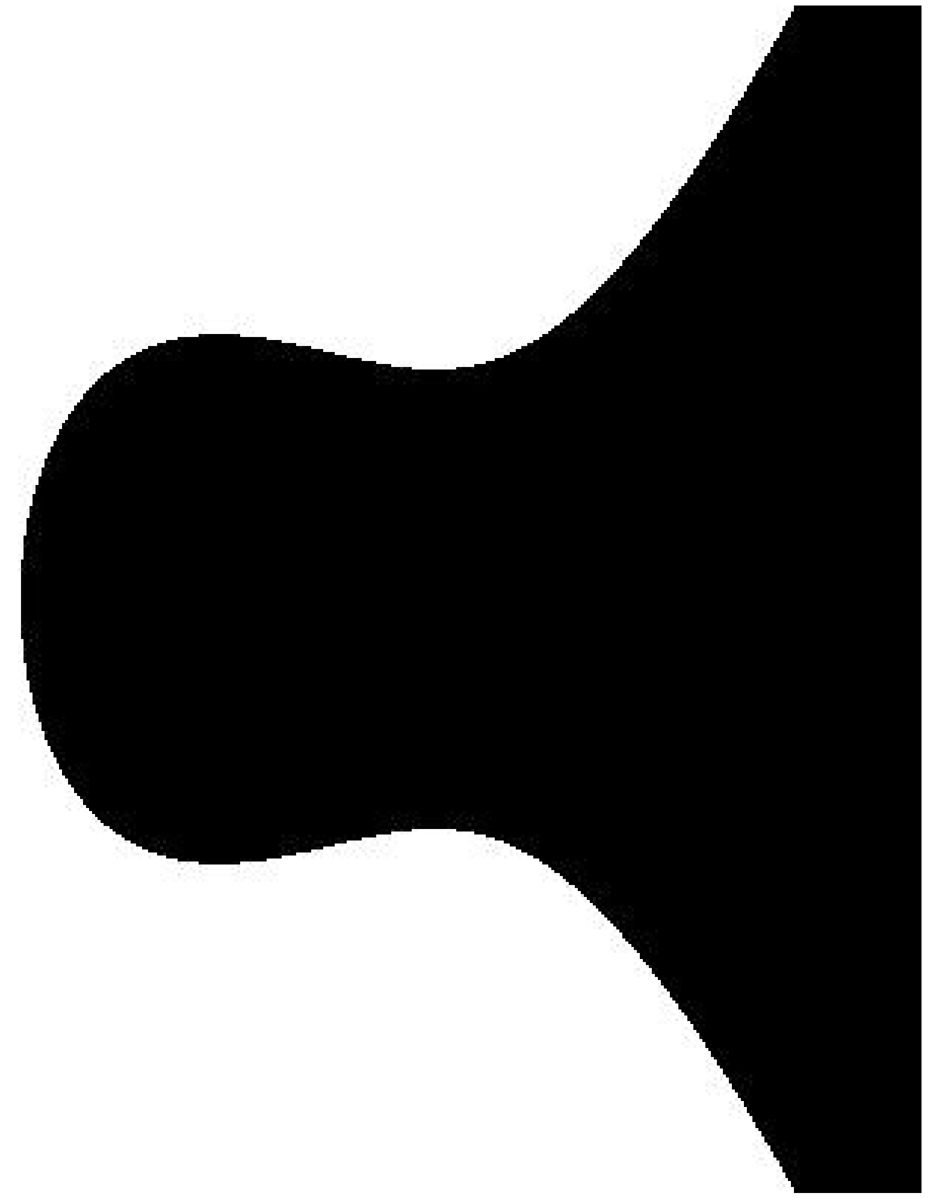


$$y^2 = x^3 - 0.4x + 0.7$$

rnstein–Lange:
or ADD, 6.2**M** for DBL.

sil–Wong–Carter–Dawson:
for ADD.





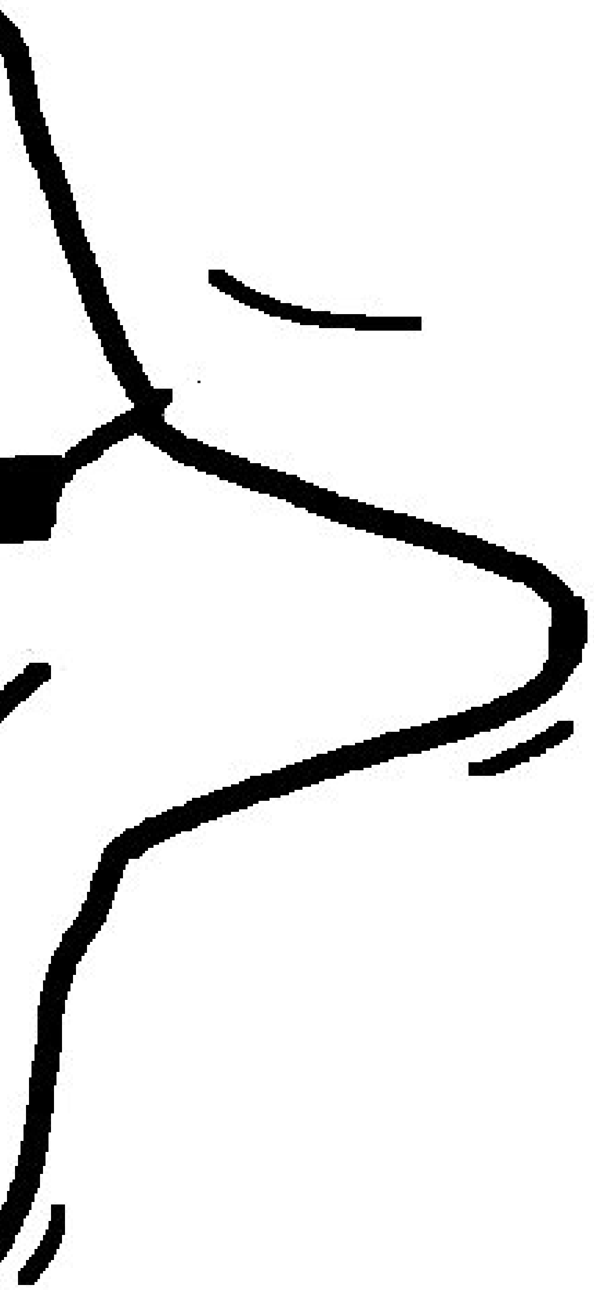
$y^2 = x^3 - 0.4x + 0.7$

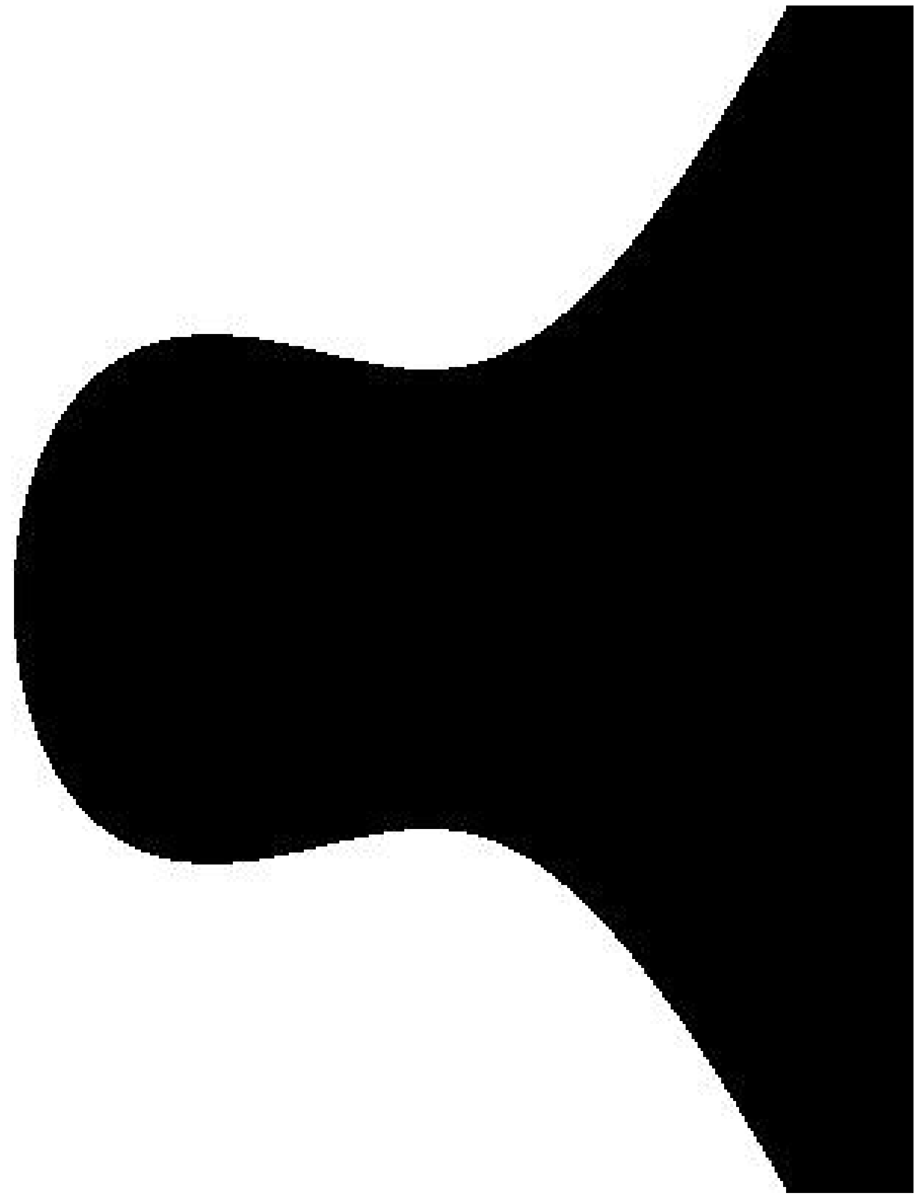
The We
turtle: o
and slow
(picture)

ange:

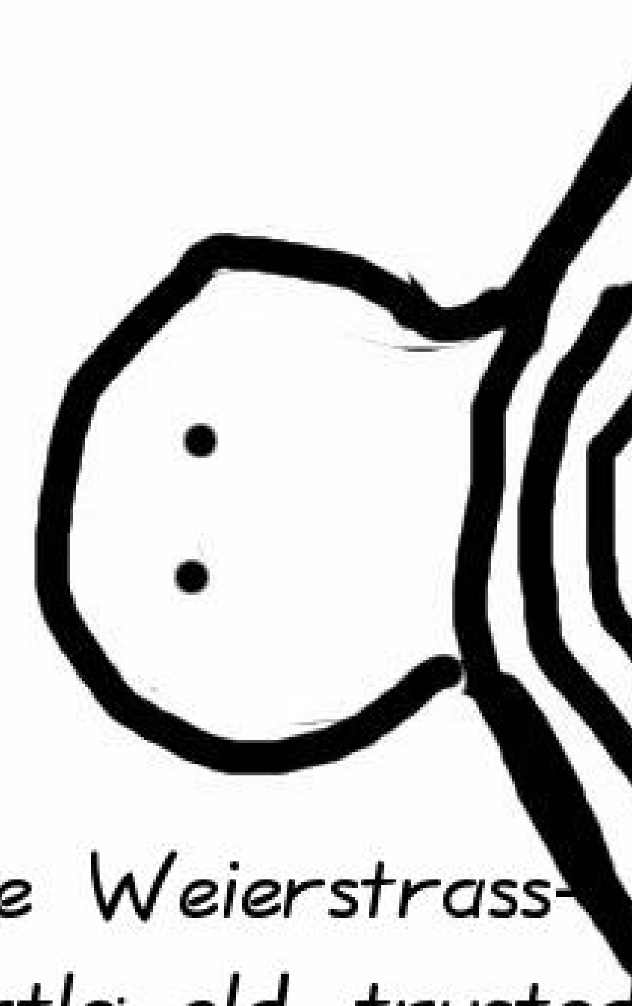
6.2**M** for DBL.

Carter–Dawson:





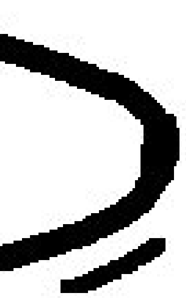
$$y^2 = x^3 - 0.4x + 0.7$$

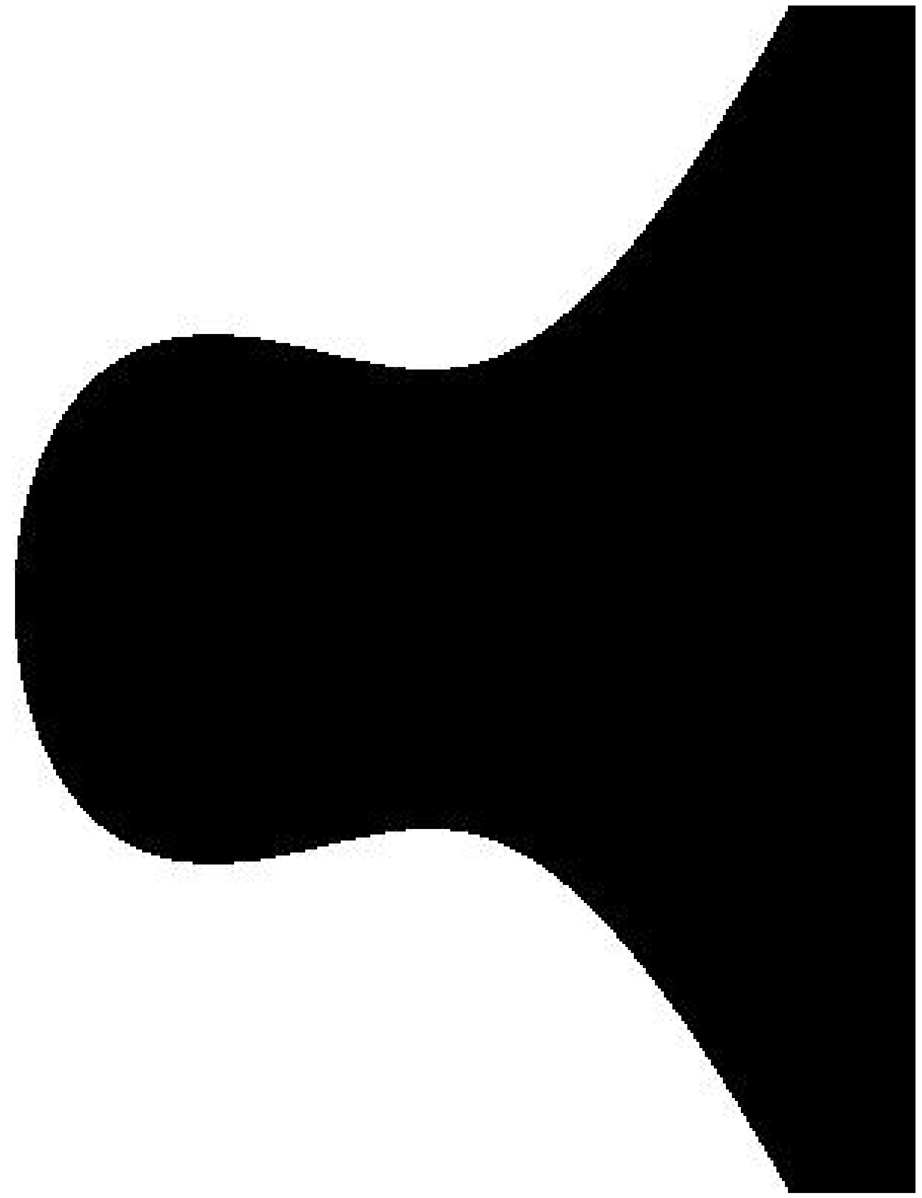


The Weierstrass-
turtle: old, trusted
and slow. Warning
(picture) incomplete

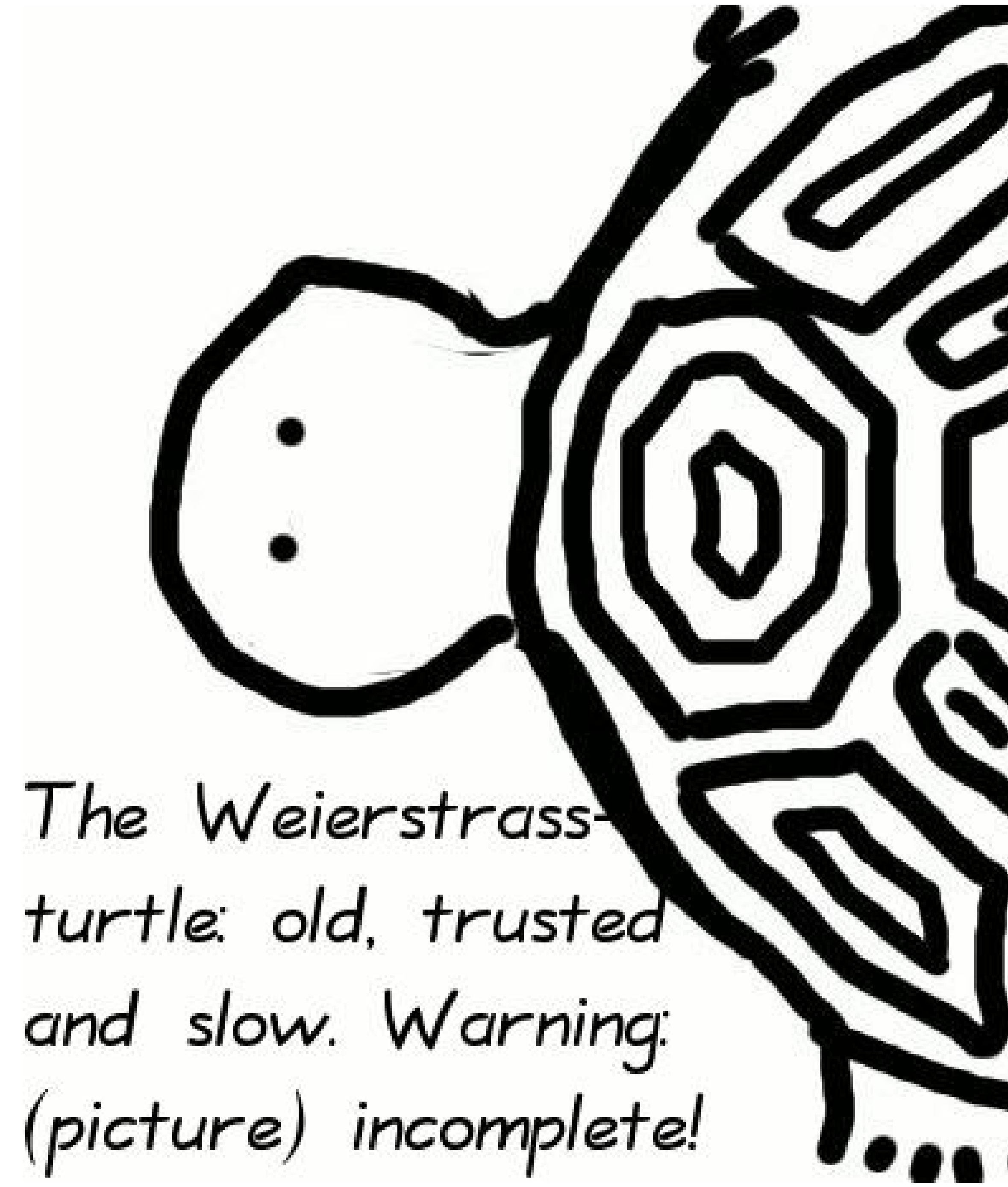
OBL.

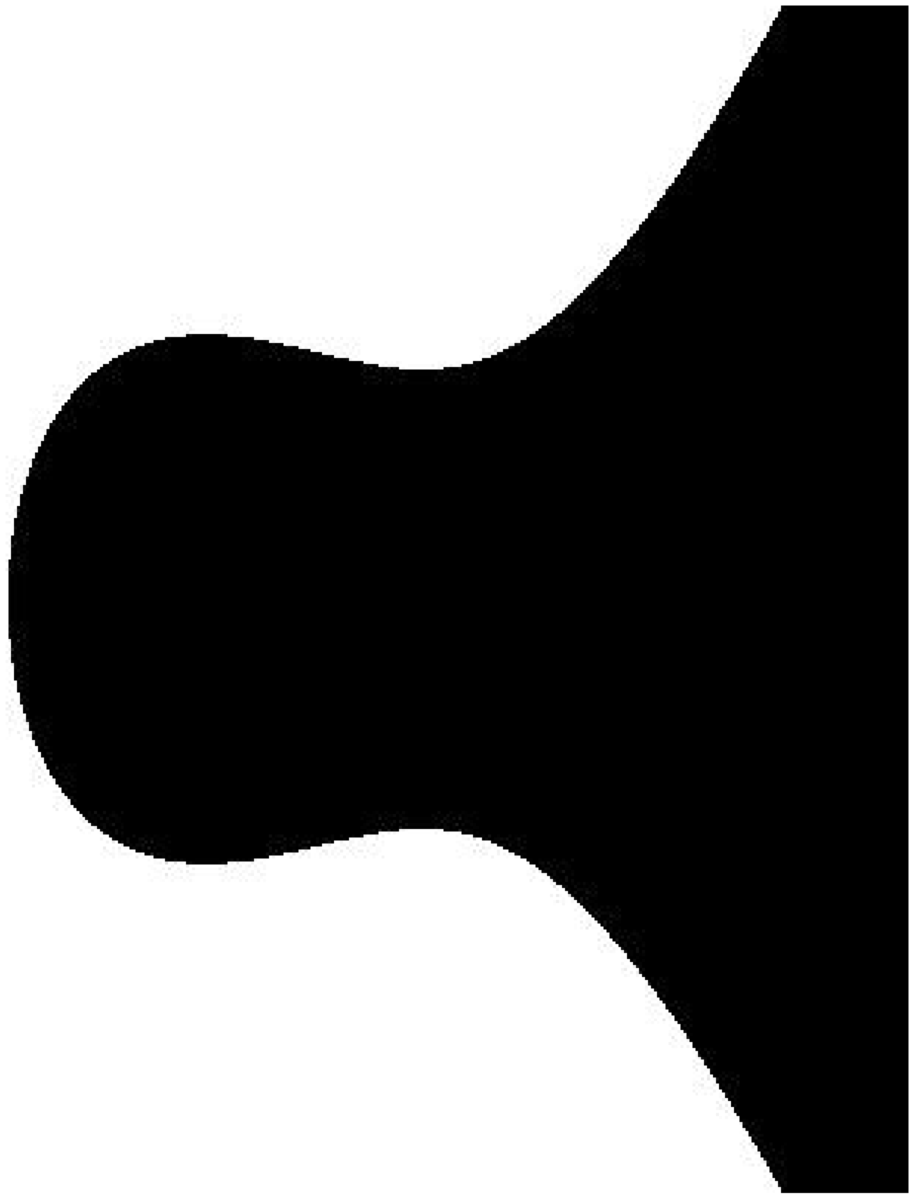
wson:



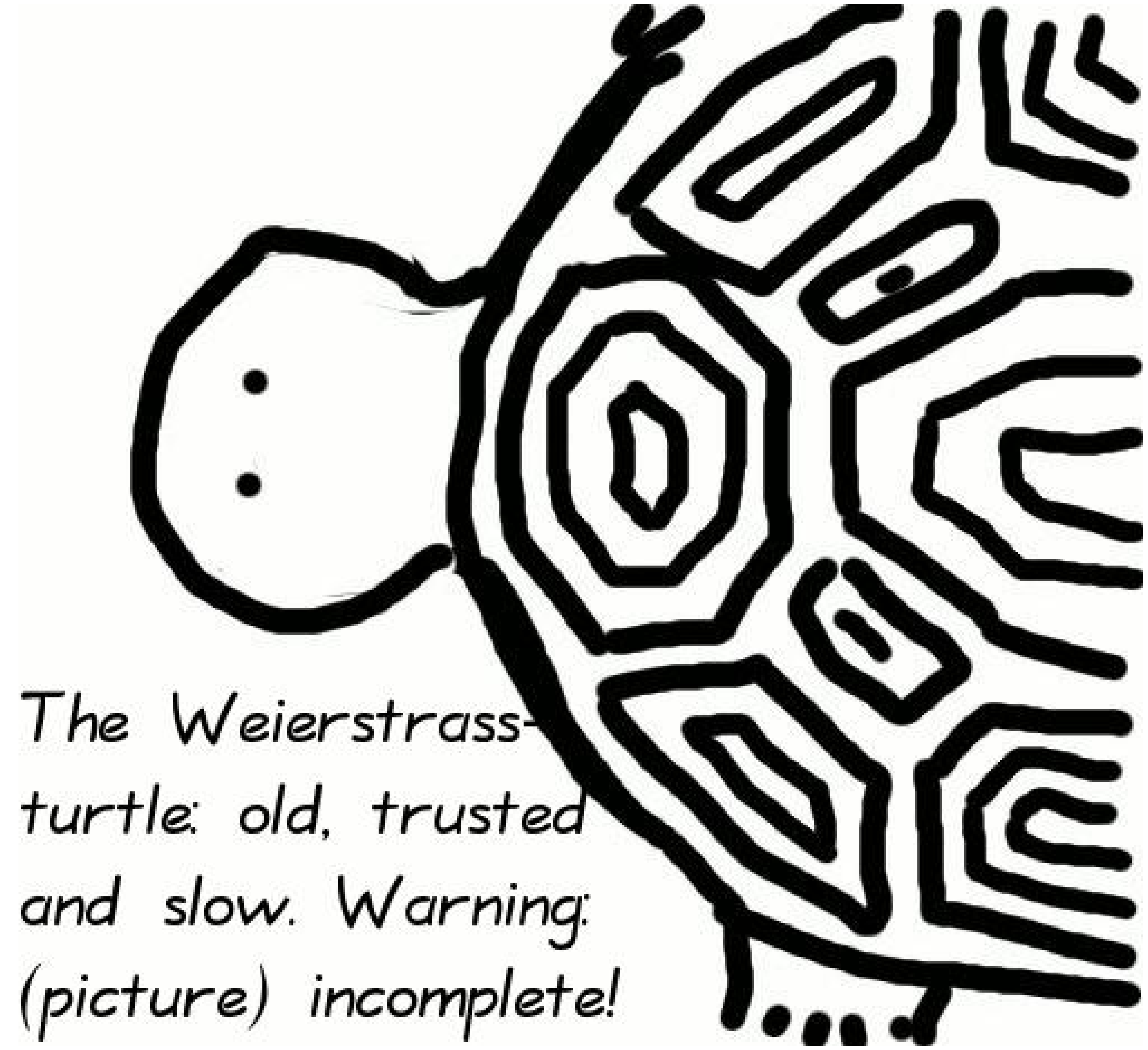


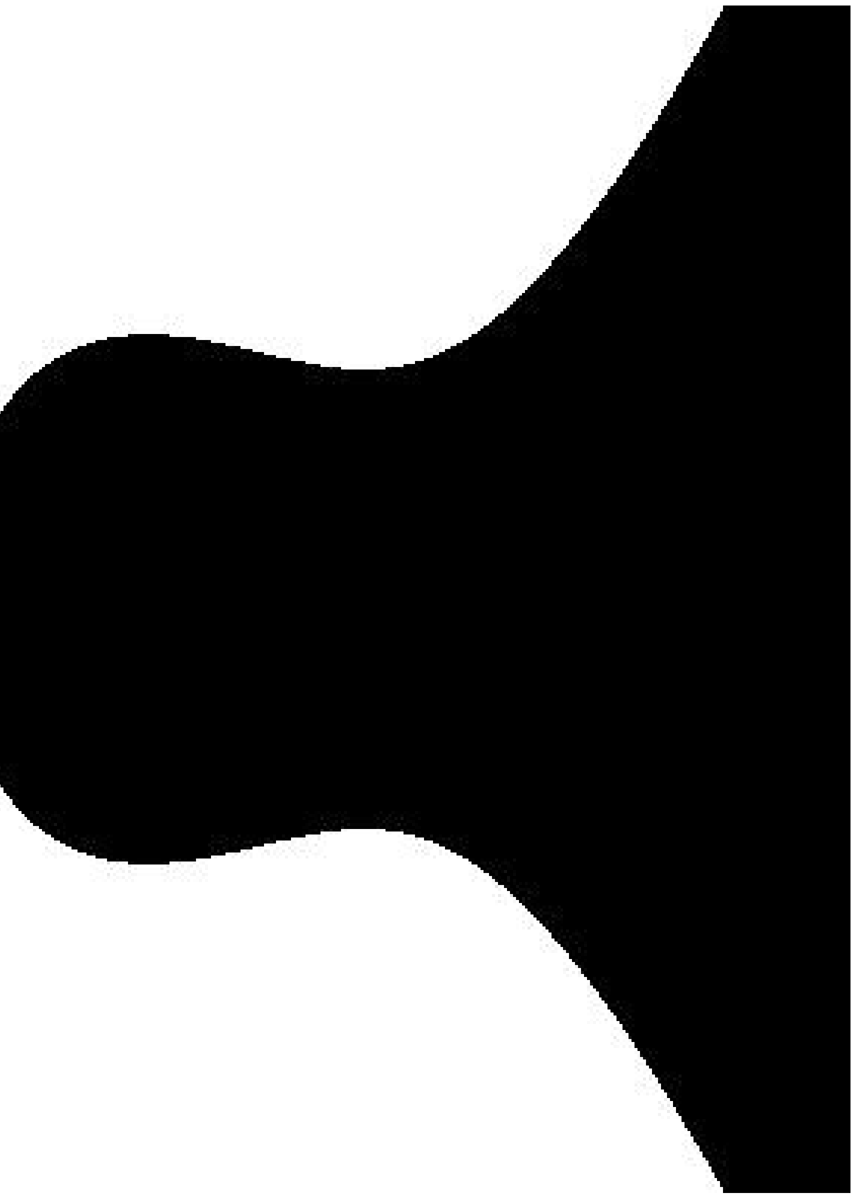
$$y^2 = x^3 - 0.4x + 0.7$$



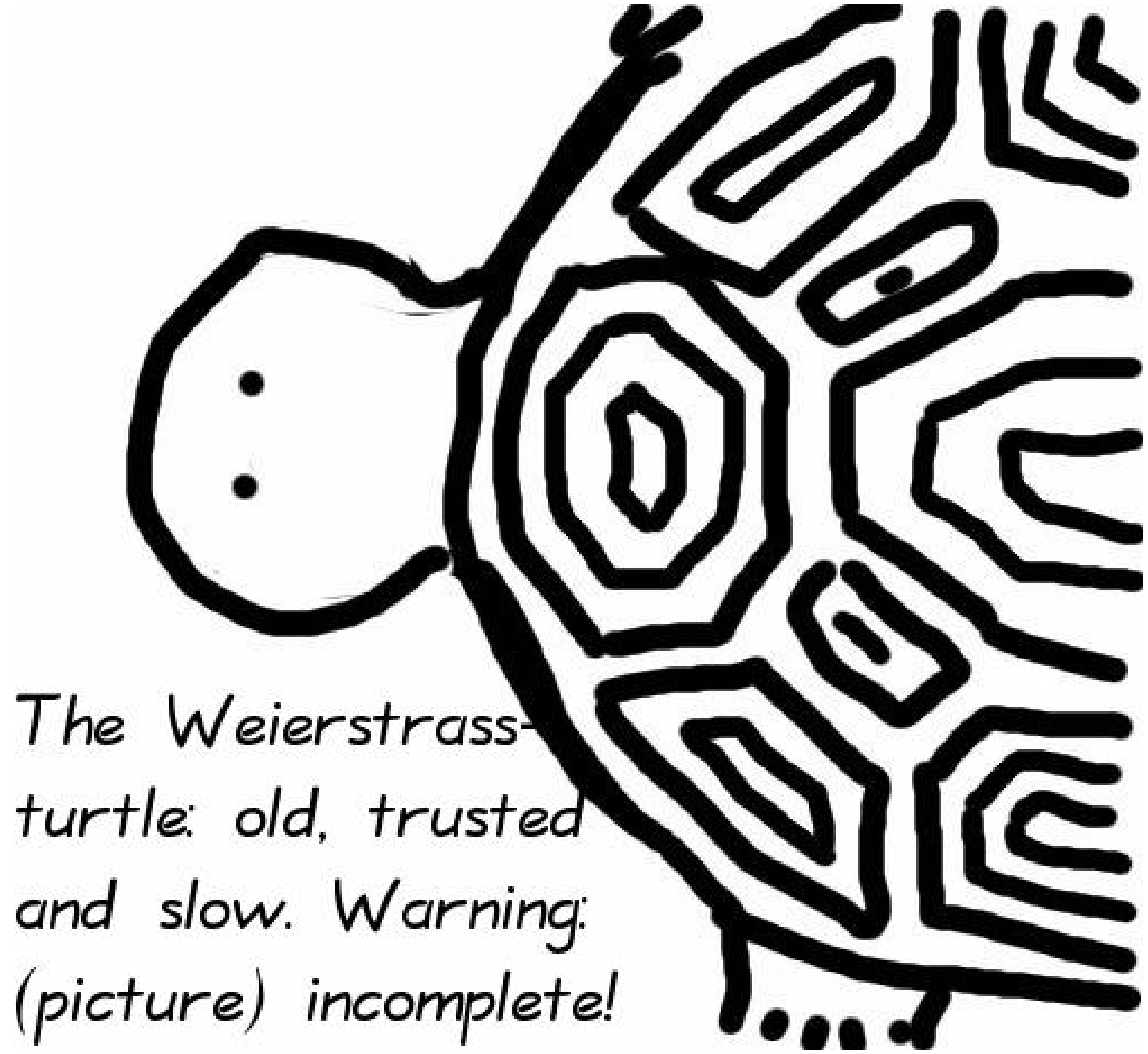


$$y^2 = x^3 - 0.4x + 0.7$$


The Weierstrass-
turtle: old, trusted
and slow. Warning:
(picture) incomplete!



$- 0.4x + 0.7$


The Weierstrass-
turtle: old, trusted
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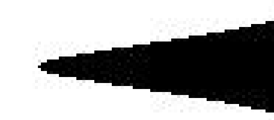
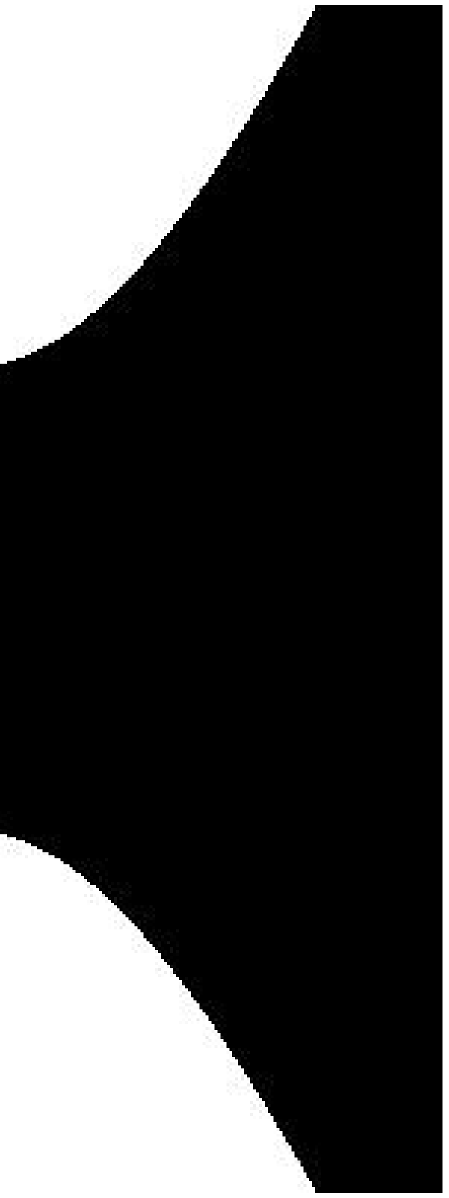
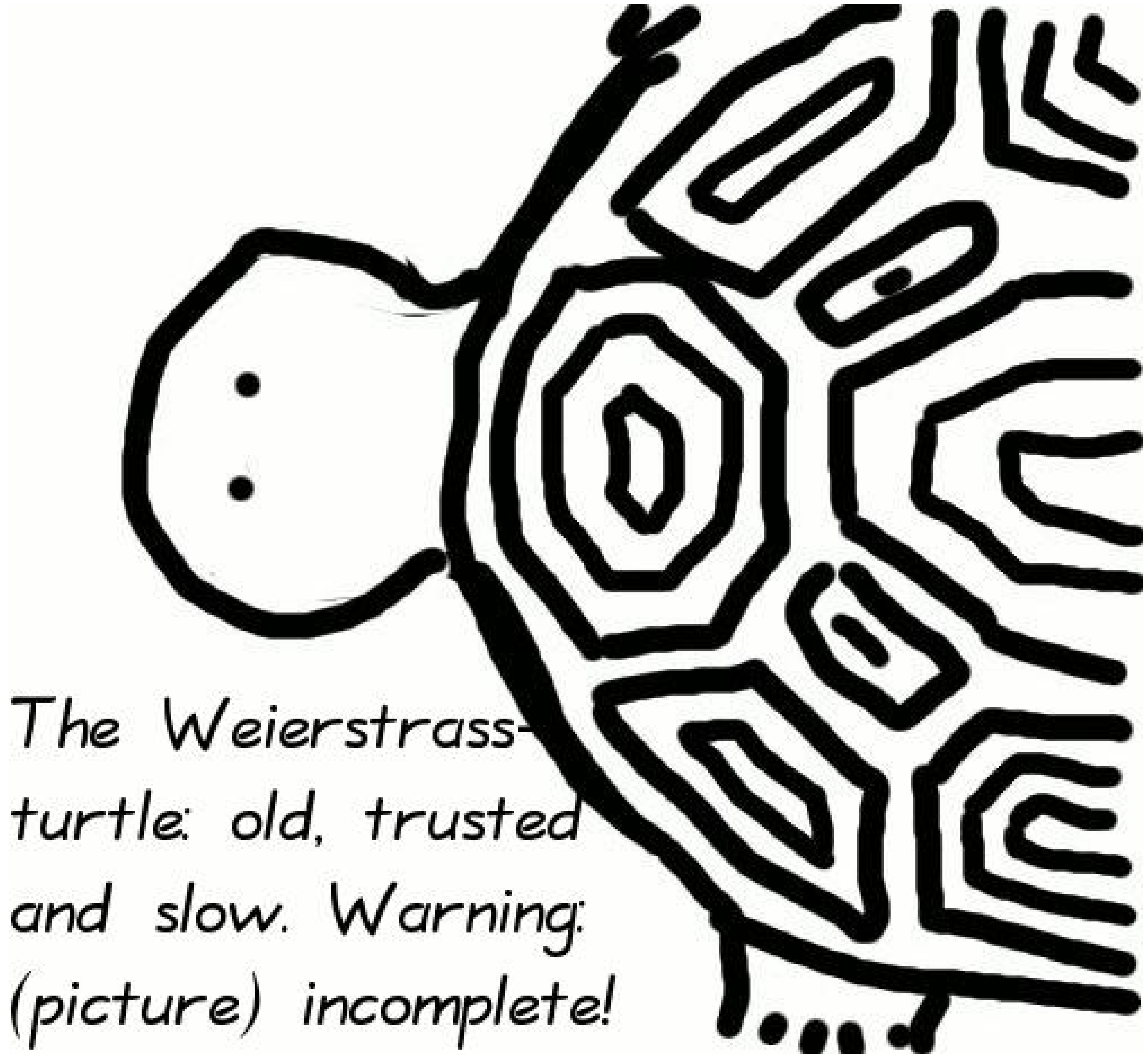
(picture) incomplete!

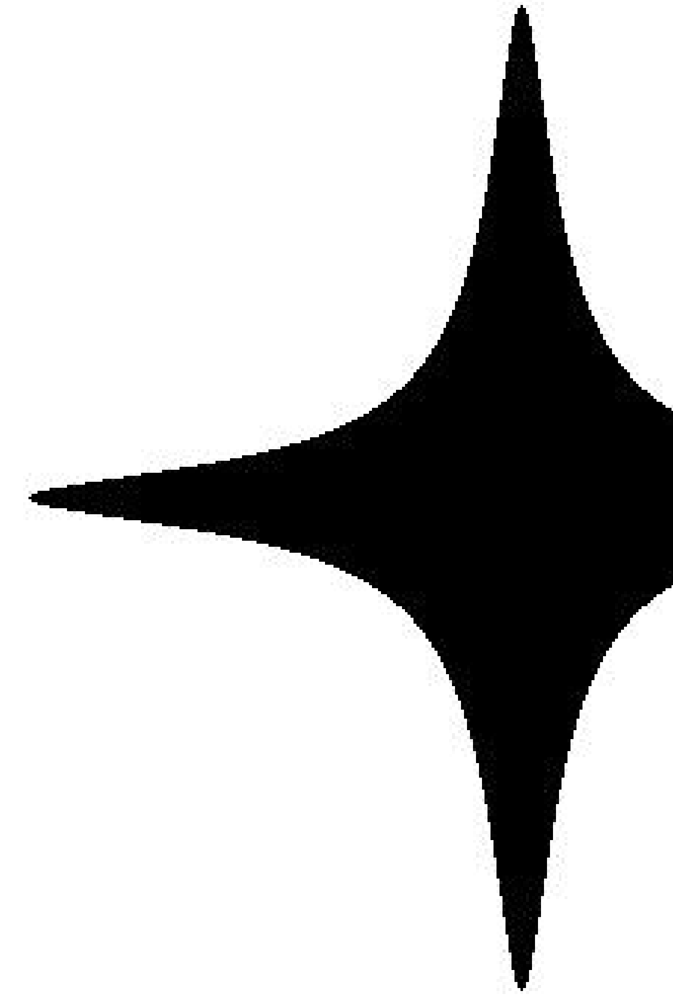


$x^2 + y^2$

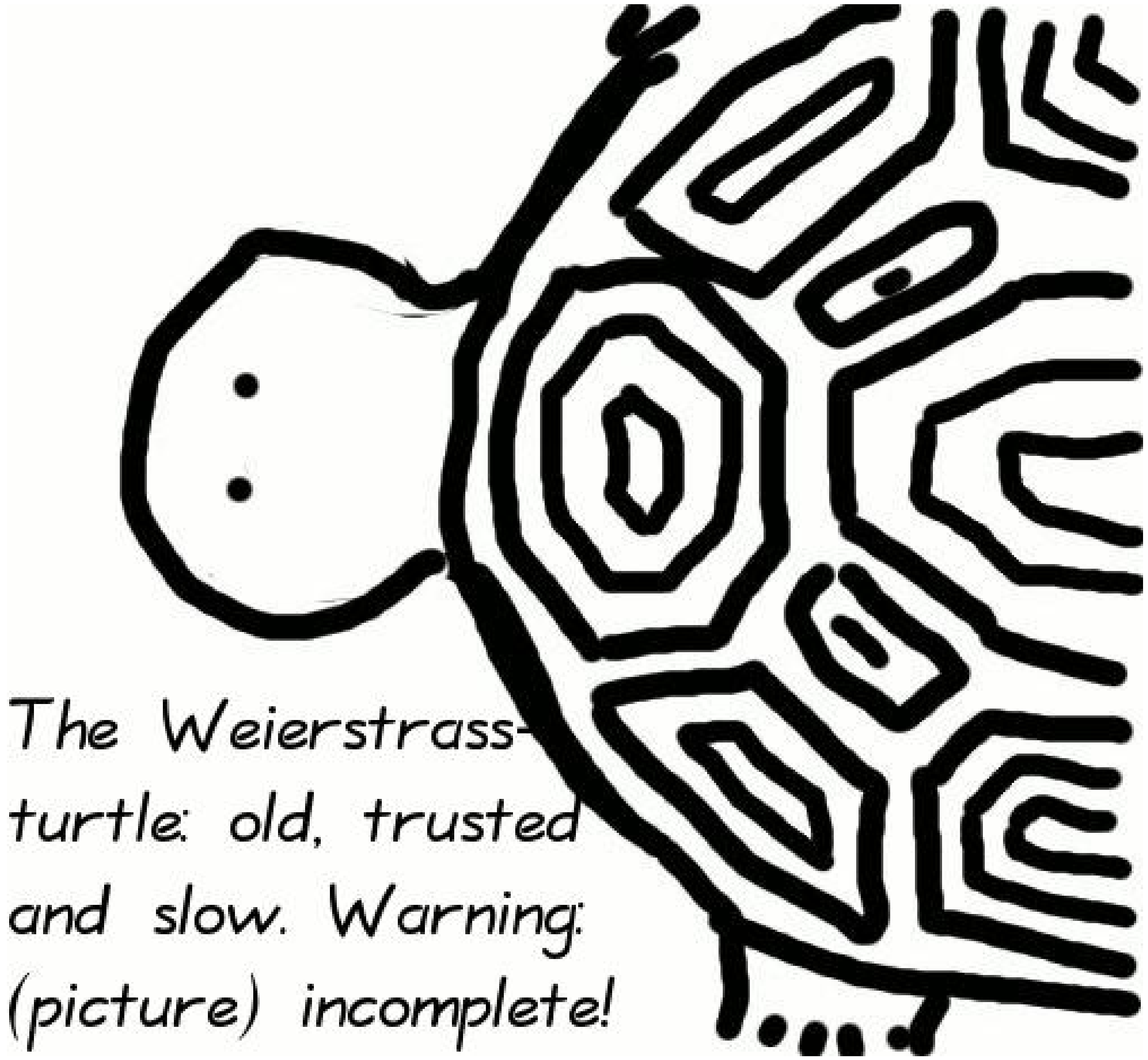


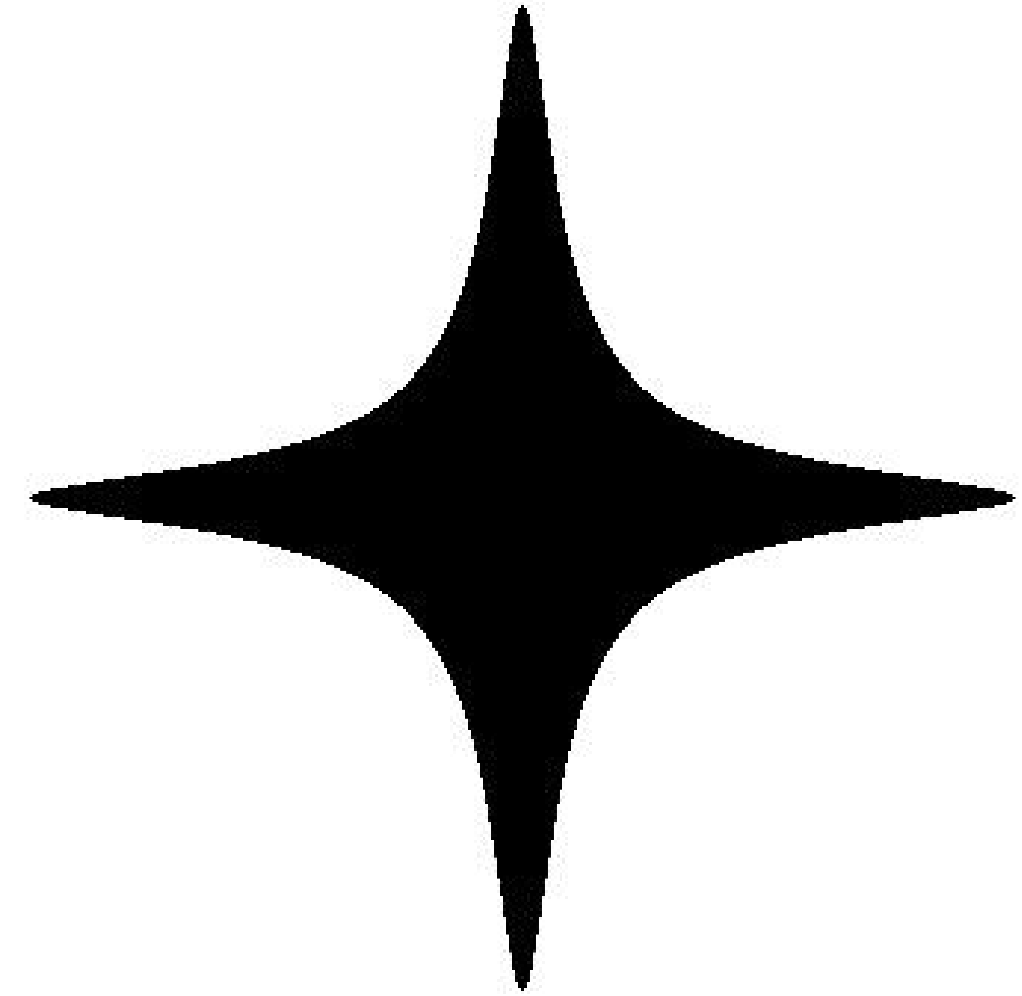
0.7


The Weierstrass-
turtle: old, trusted
and slow. Warning:
(picture) incomplete!

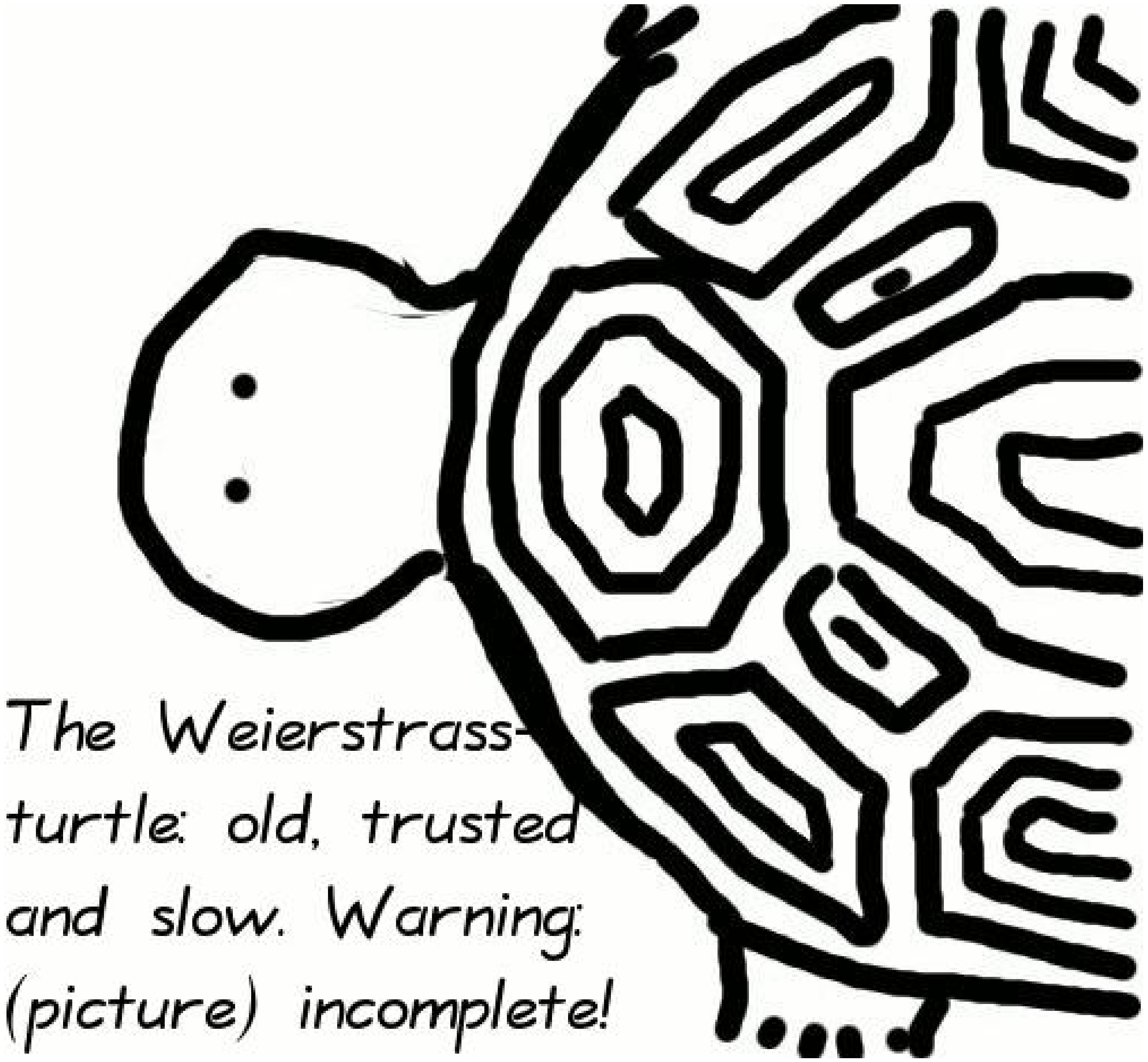


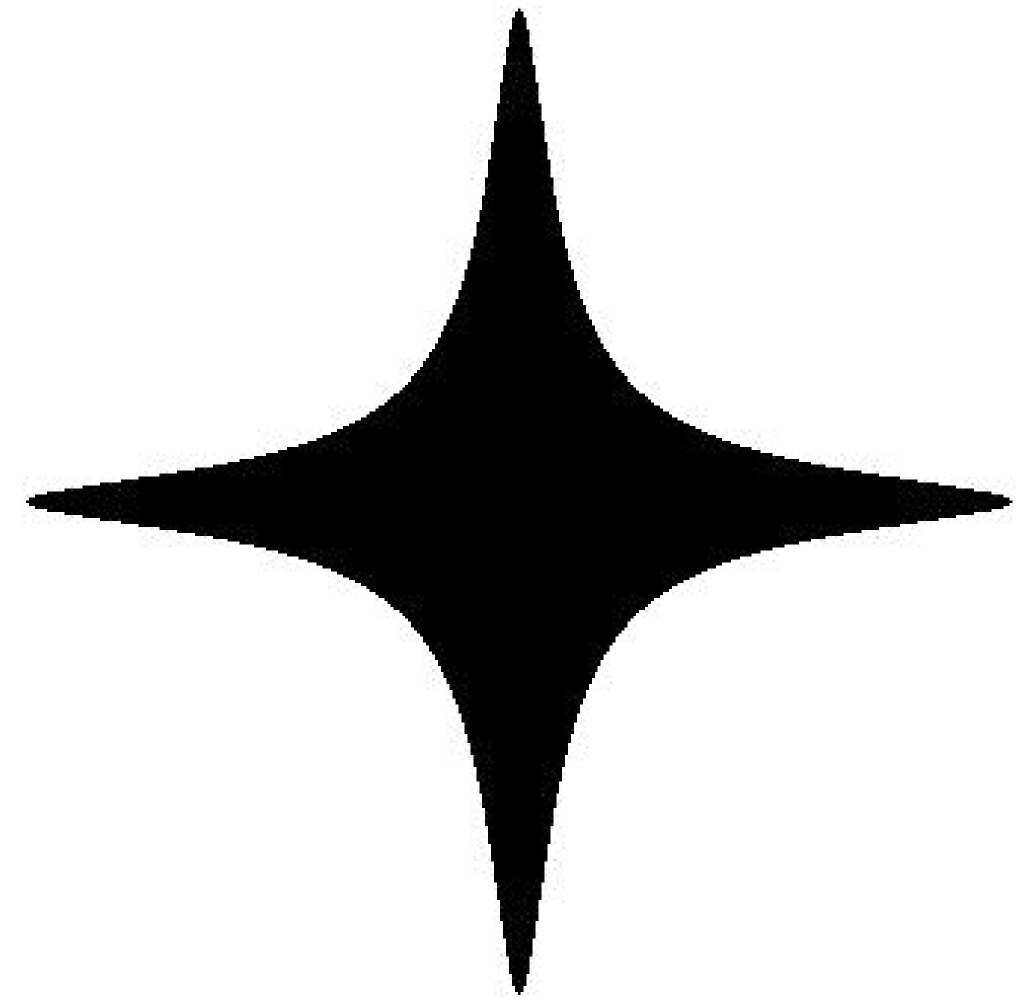
$x^2 + y^2 = 1 - 300$





$$x^2 + y^2 = 1 - 300x^2y^2$$





$$x^2 + y^2 = 1 - 300x^2y^2$$

ierstrass-
ld, trusted
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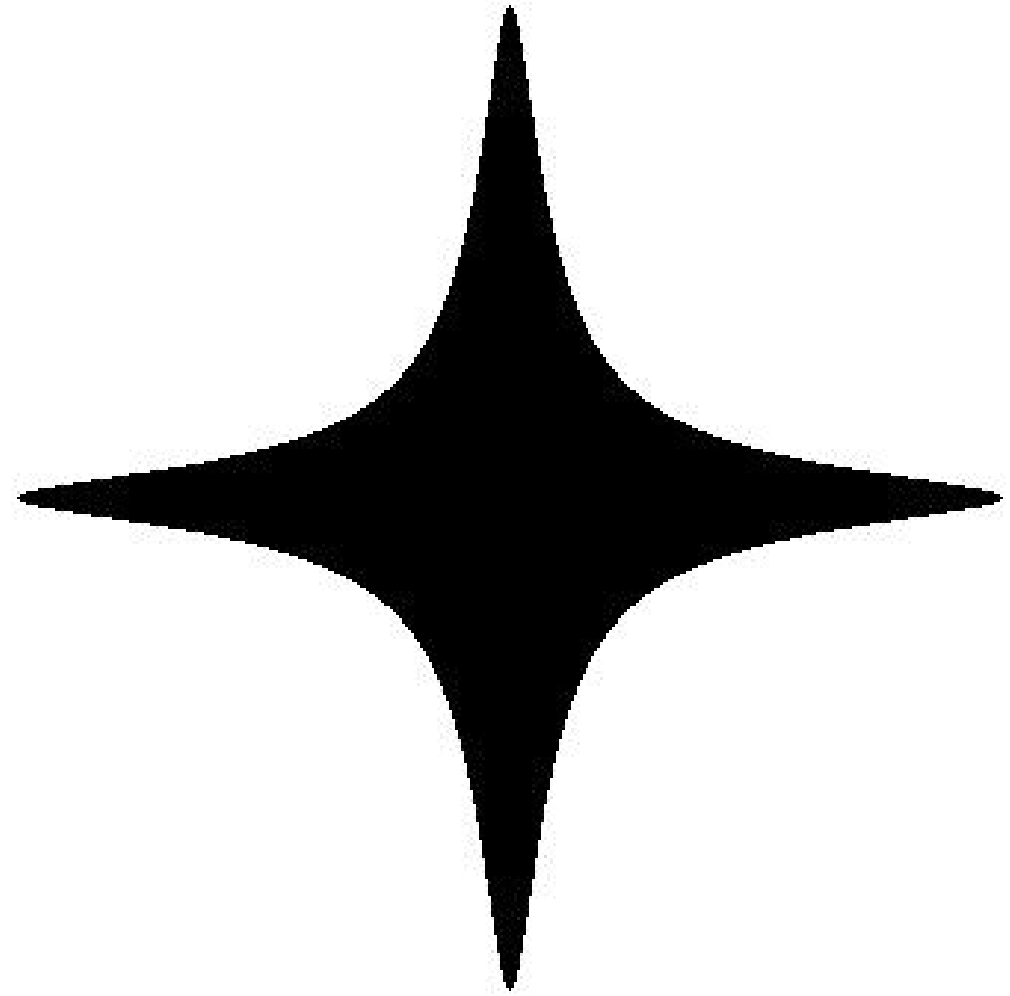
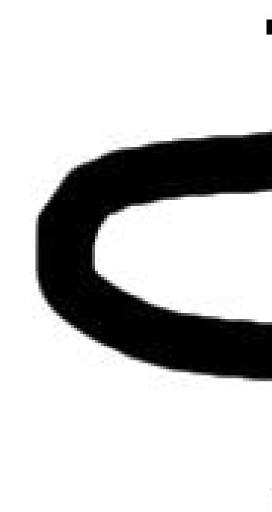
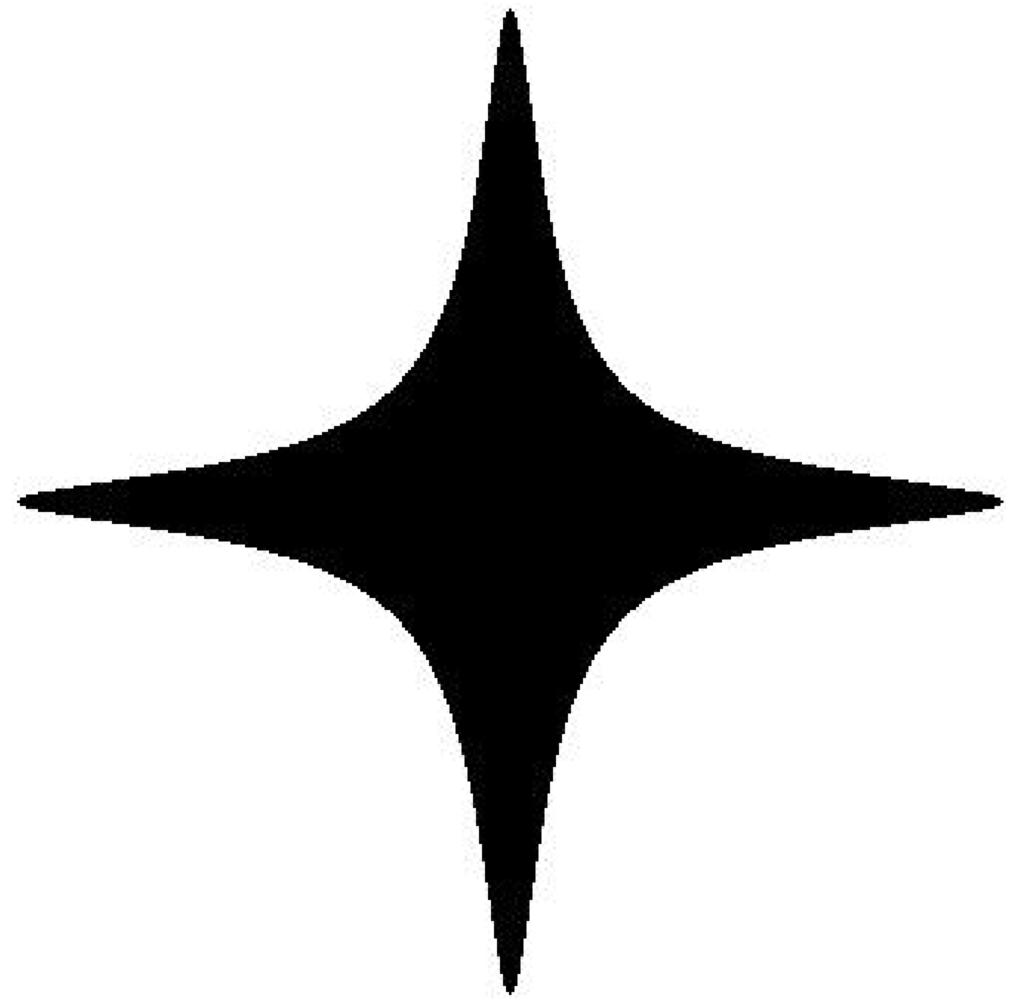


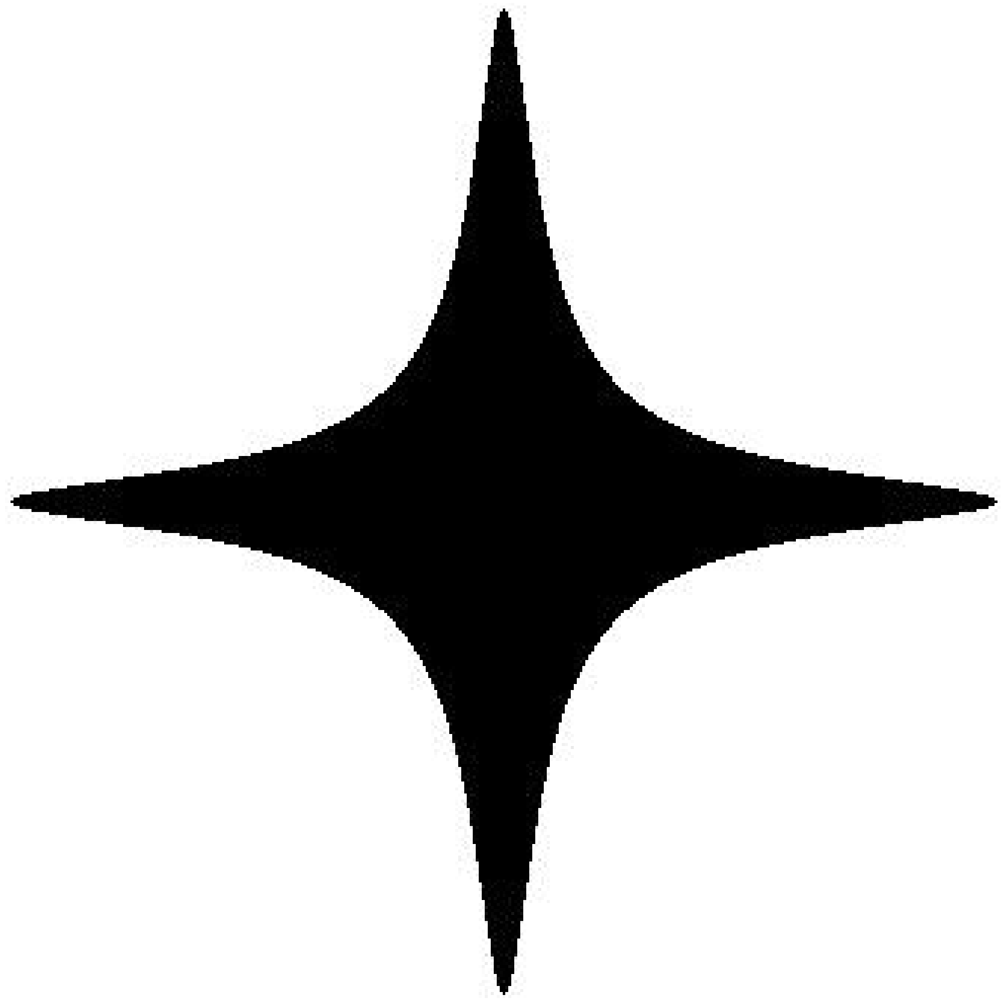
$$x^2 + y^2 = 1 - 300x^2y^2$$



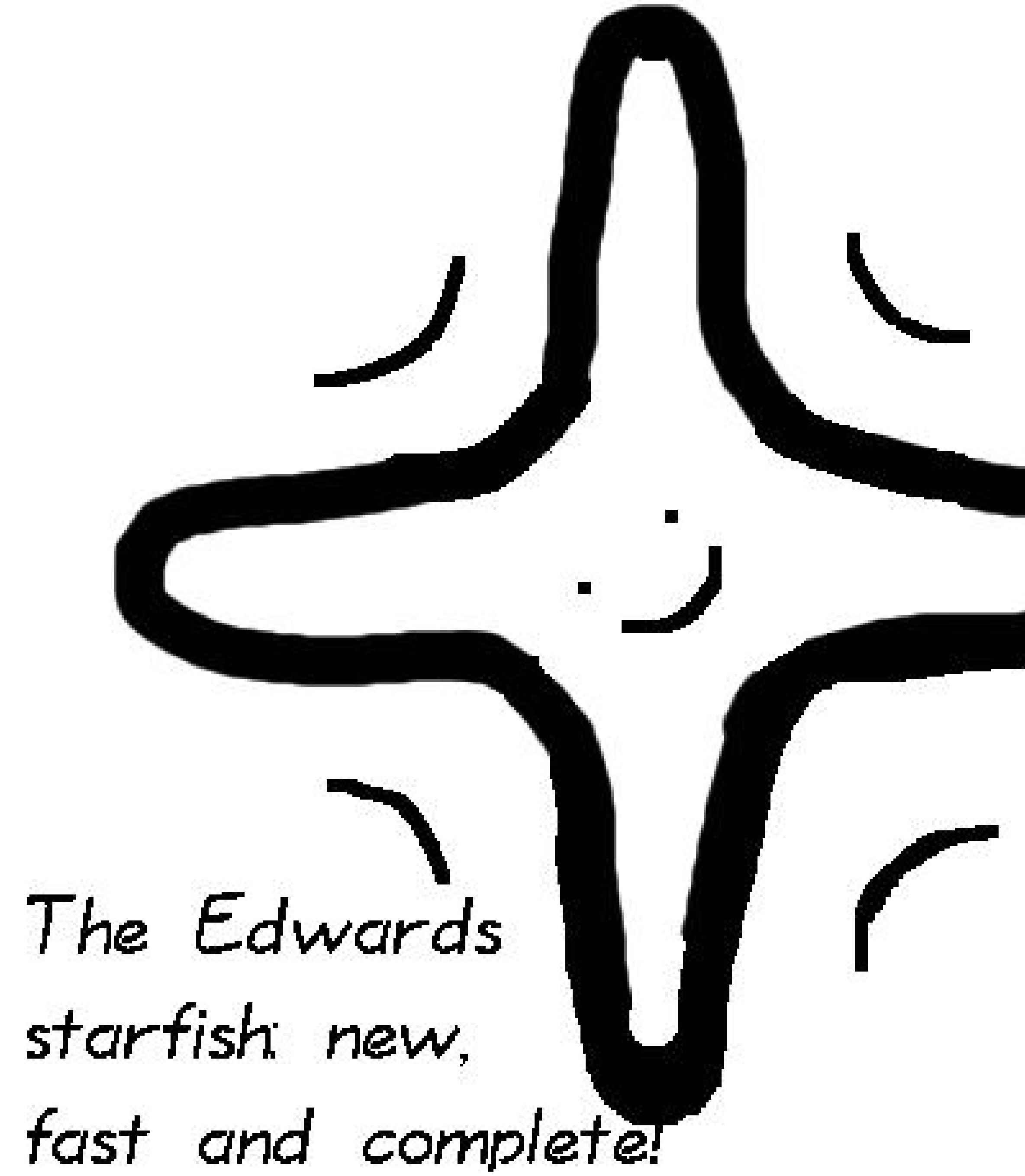
The Edv
starfish:
fast and

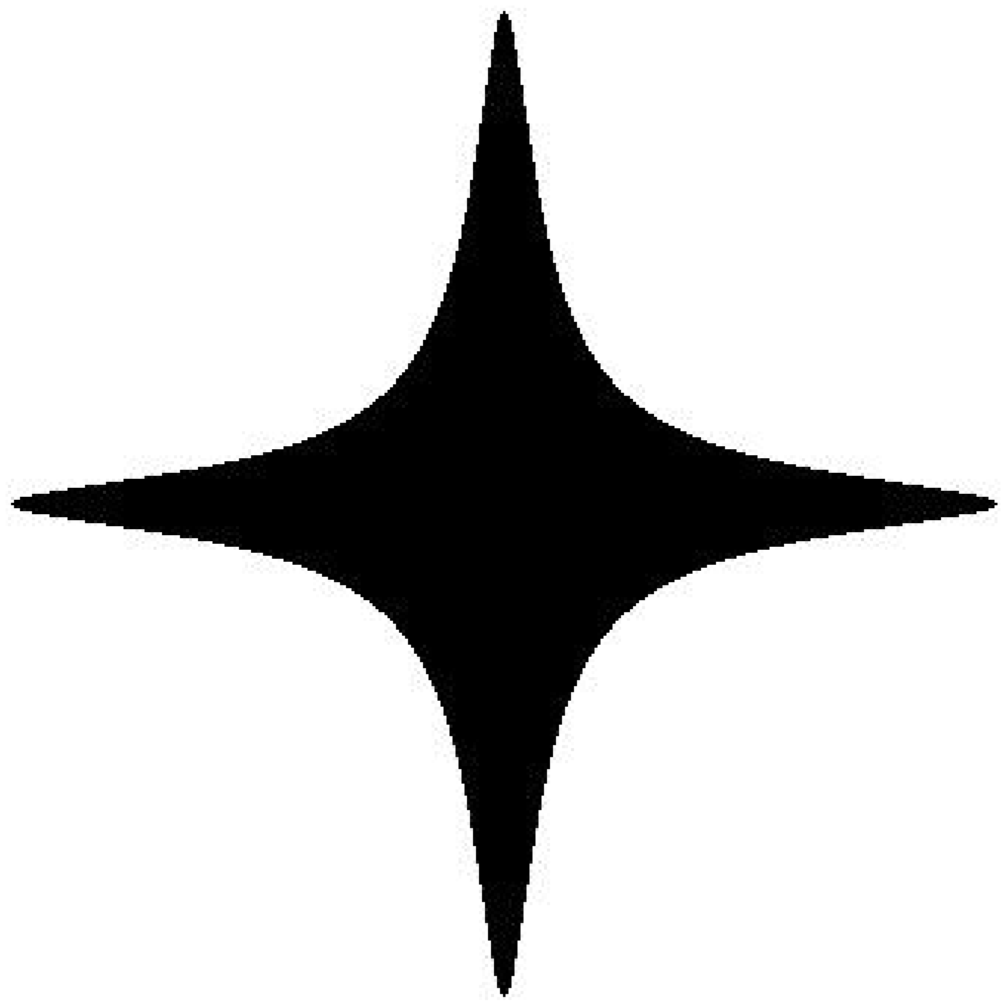


$$x^2 + y^2 = 1 - 300x^2y^2$$

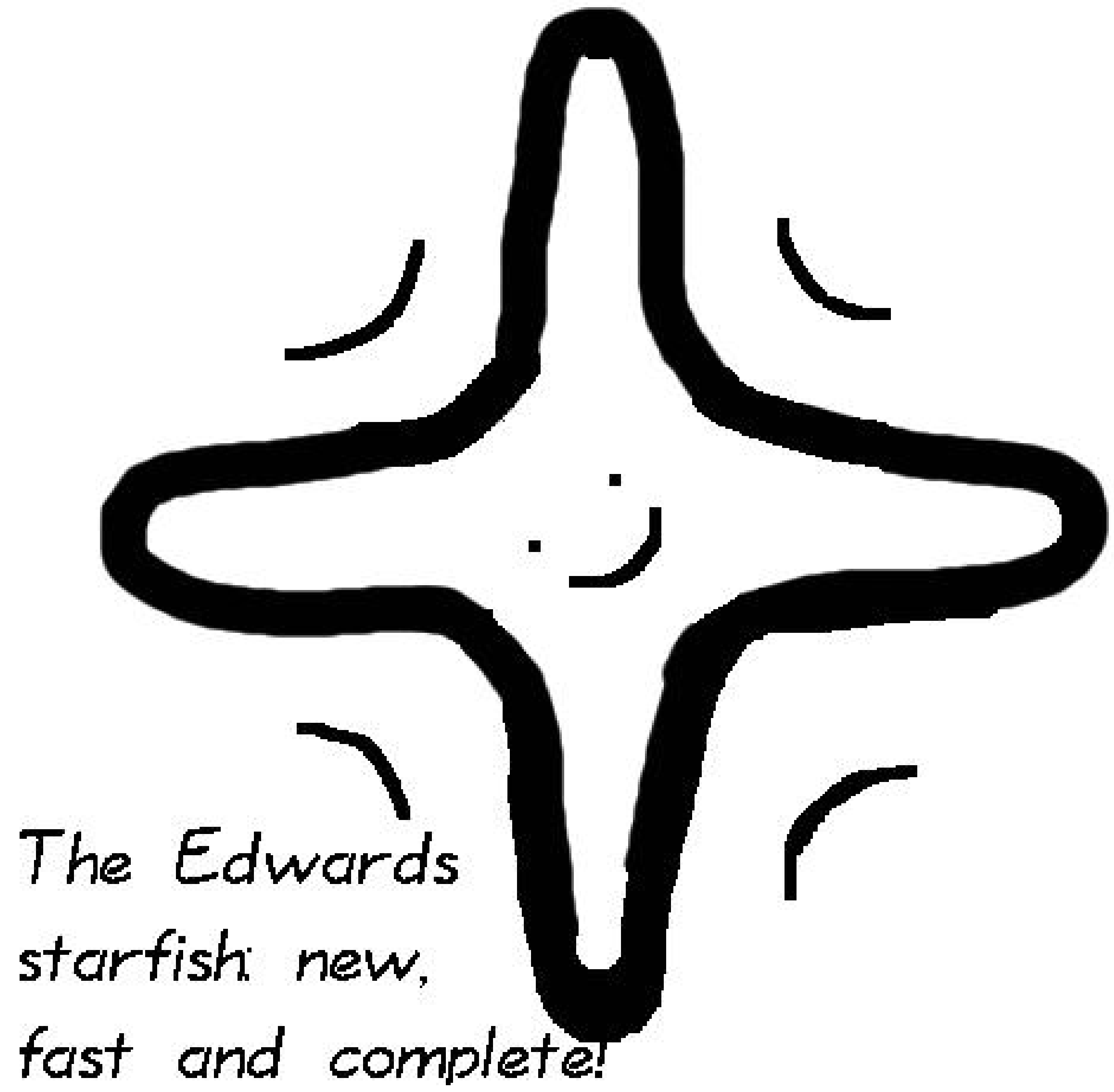


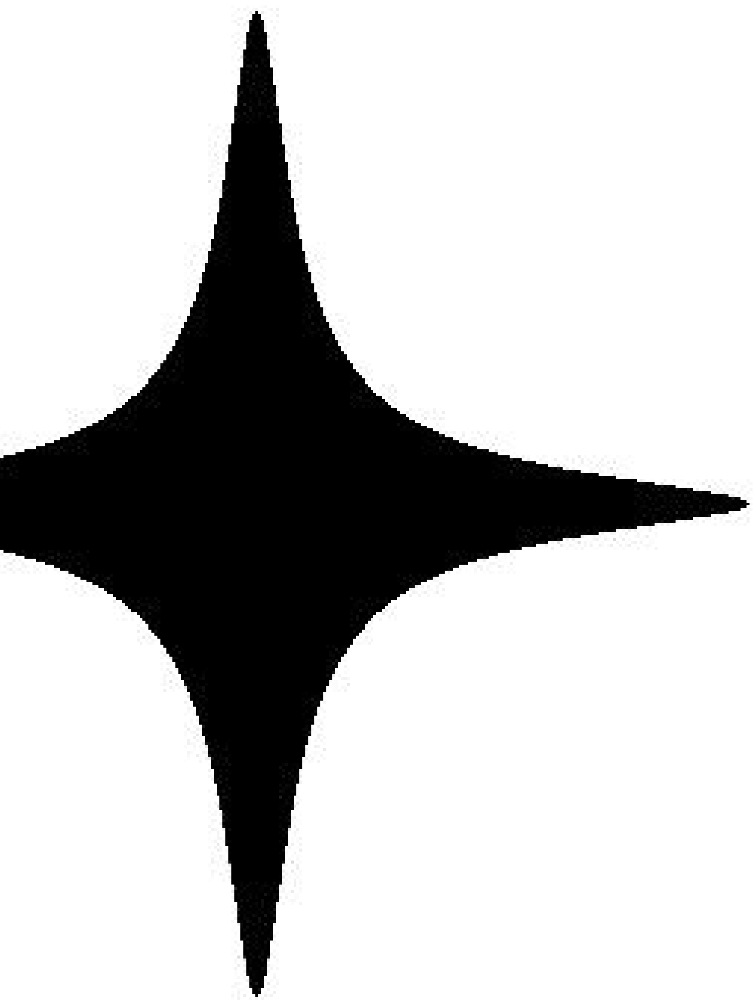
$$x^2 + y^2 = 1 - 300x^2y^2$$



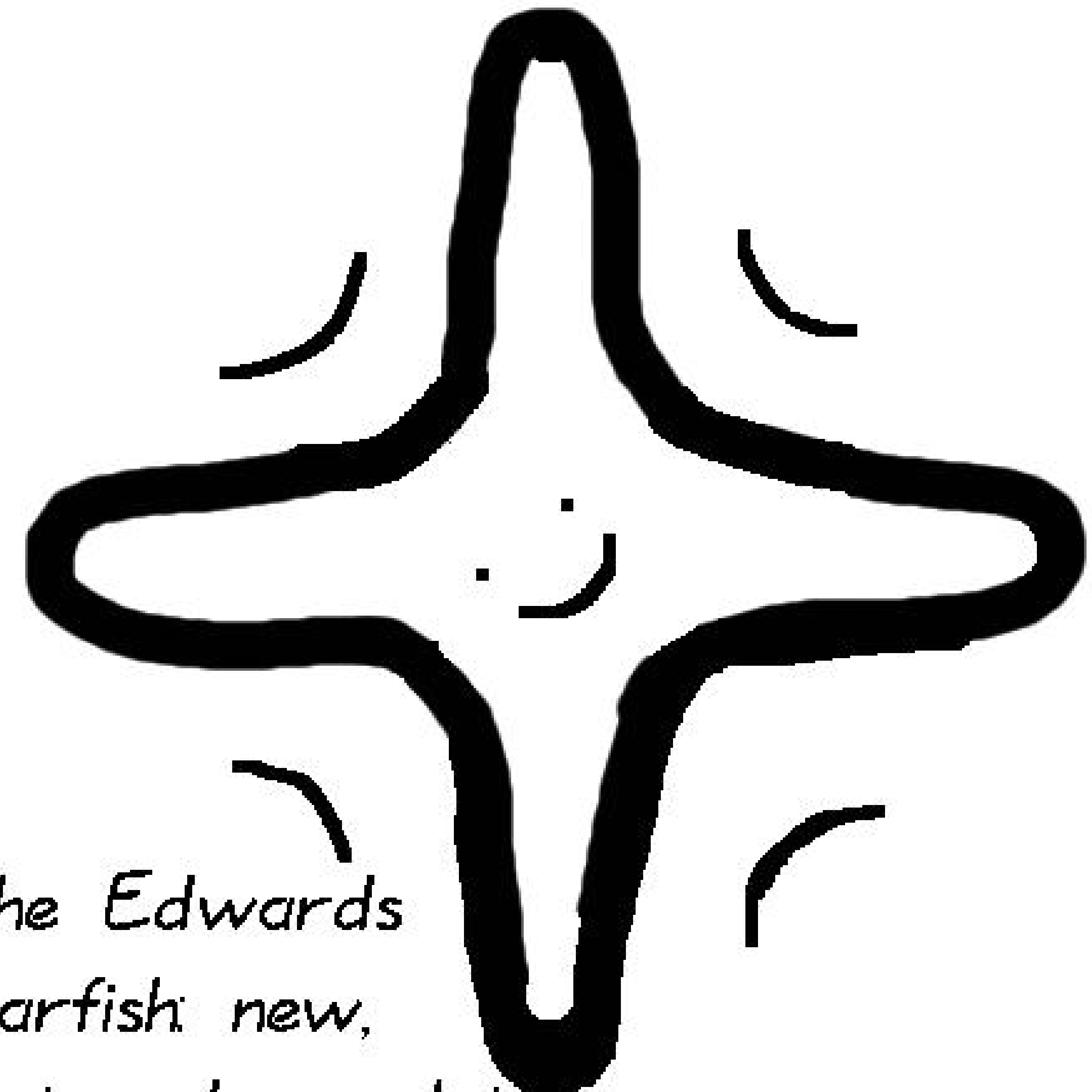


$$x^2 + y^2 = 1 - 300x^2y^2$$

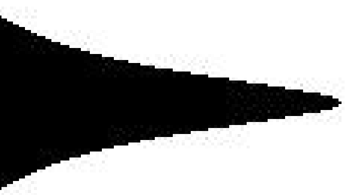




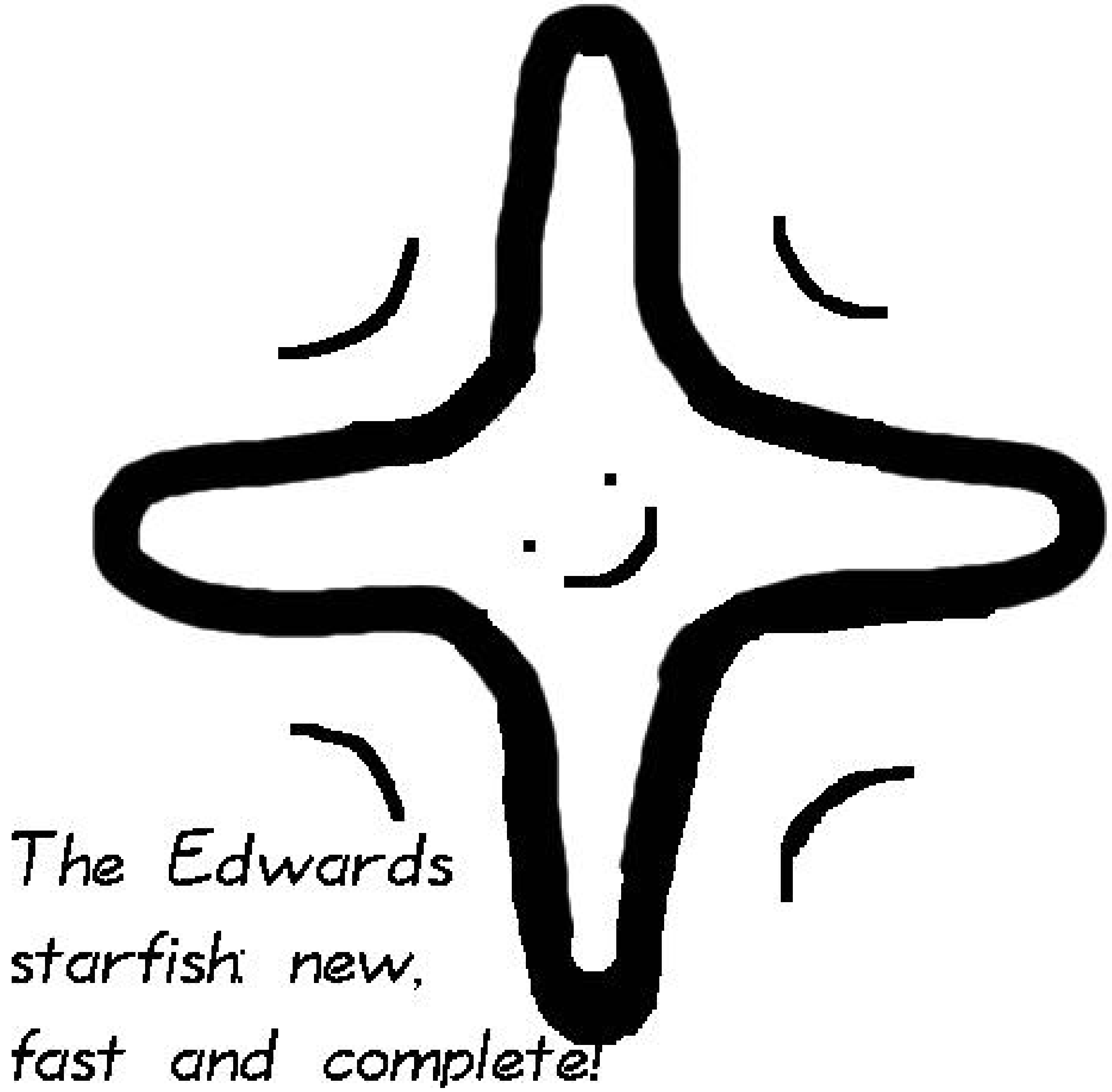
$= 1 - 300x^2y^2$

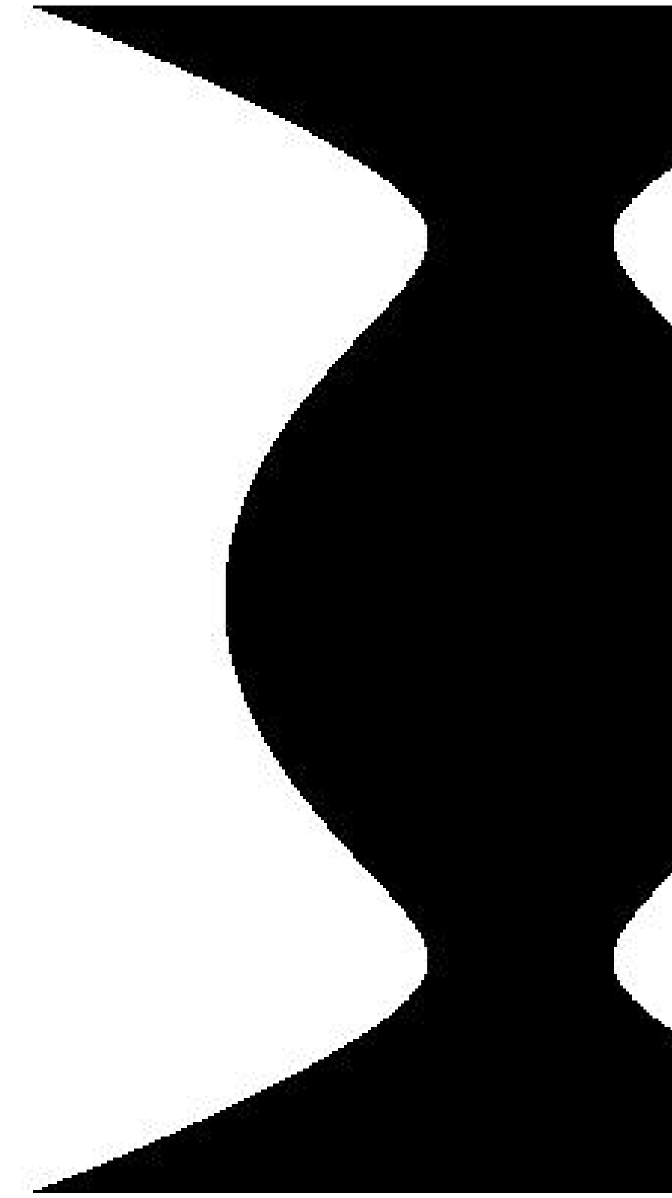


$x^2 = y^4$

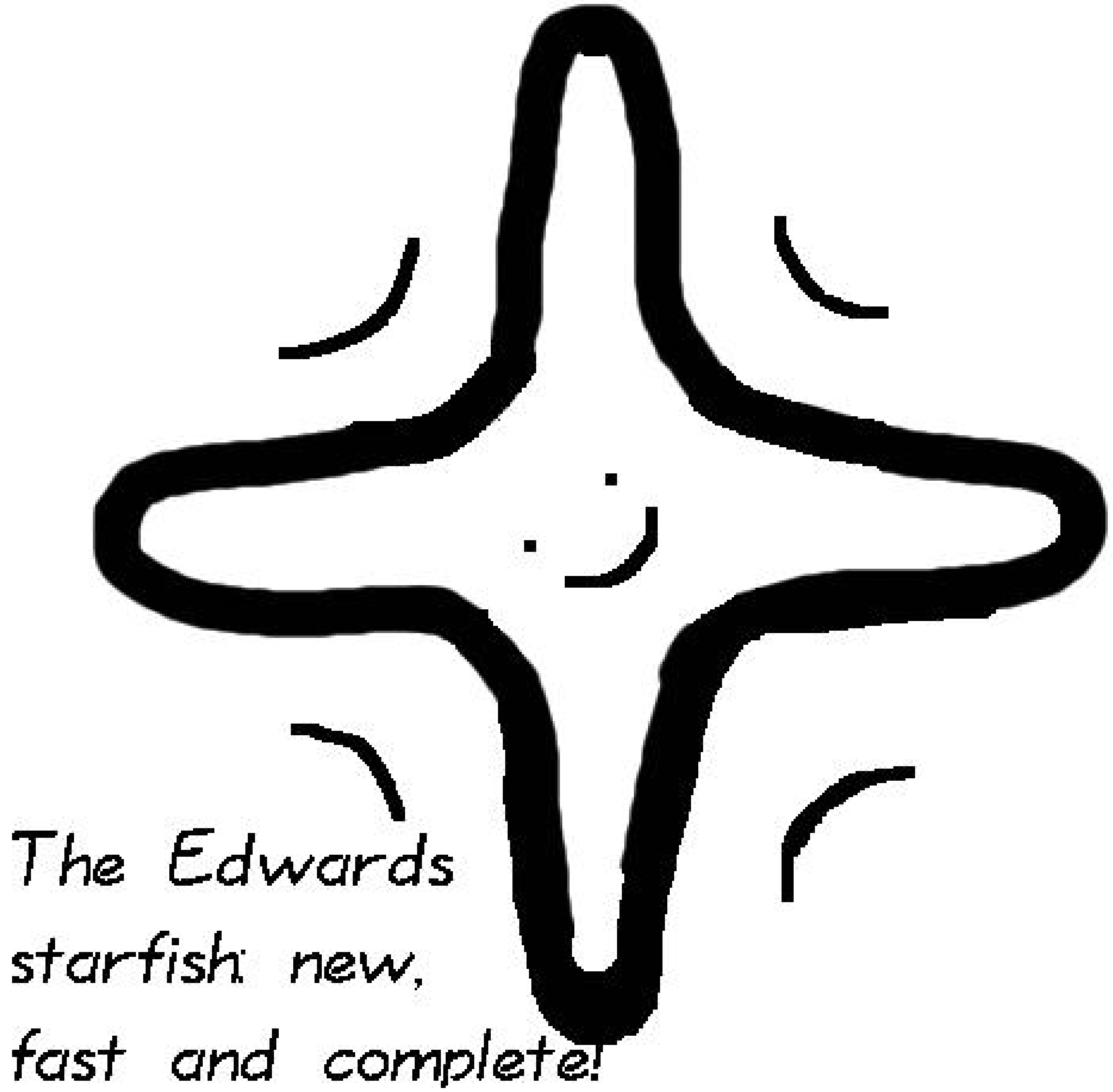


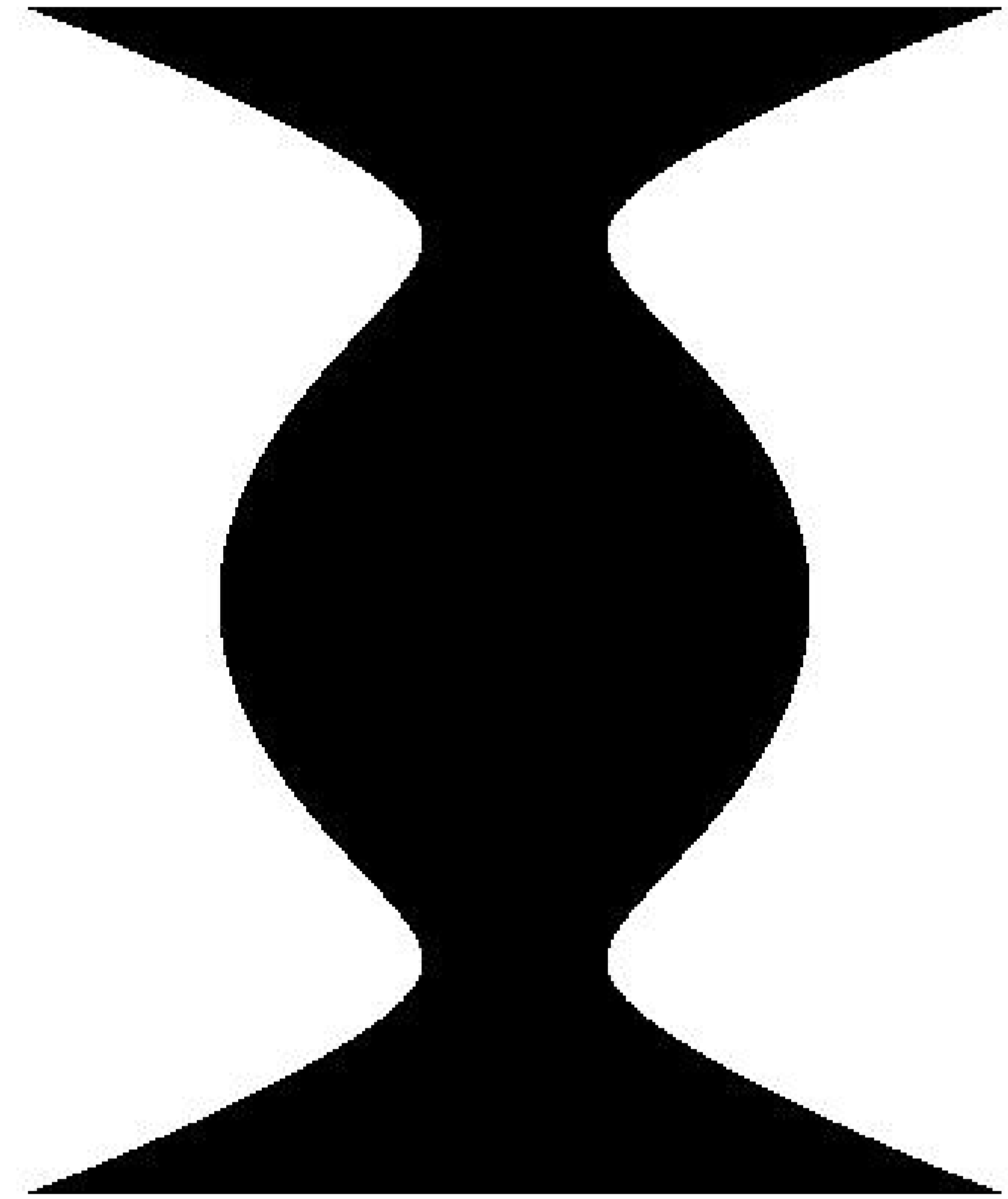
$0x^2y^2$



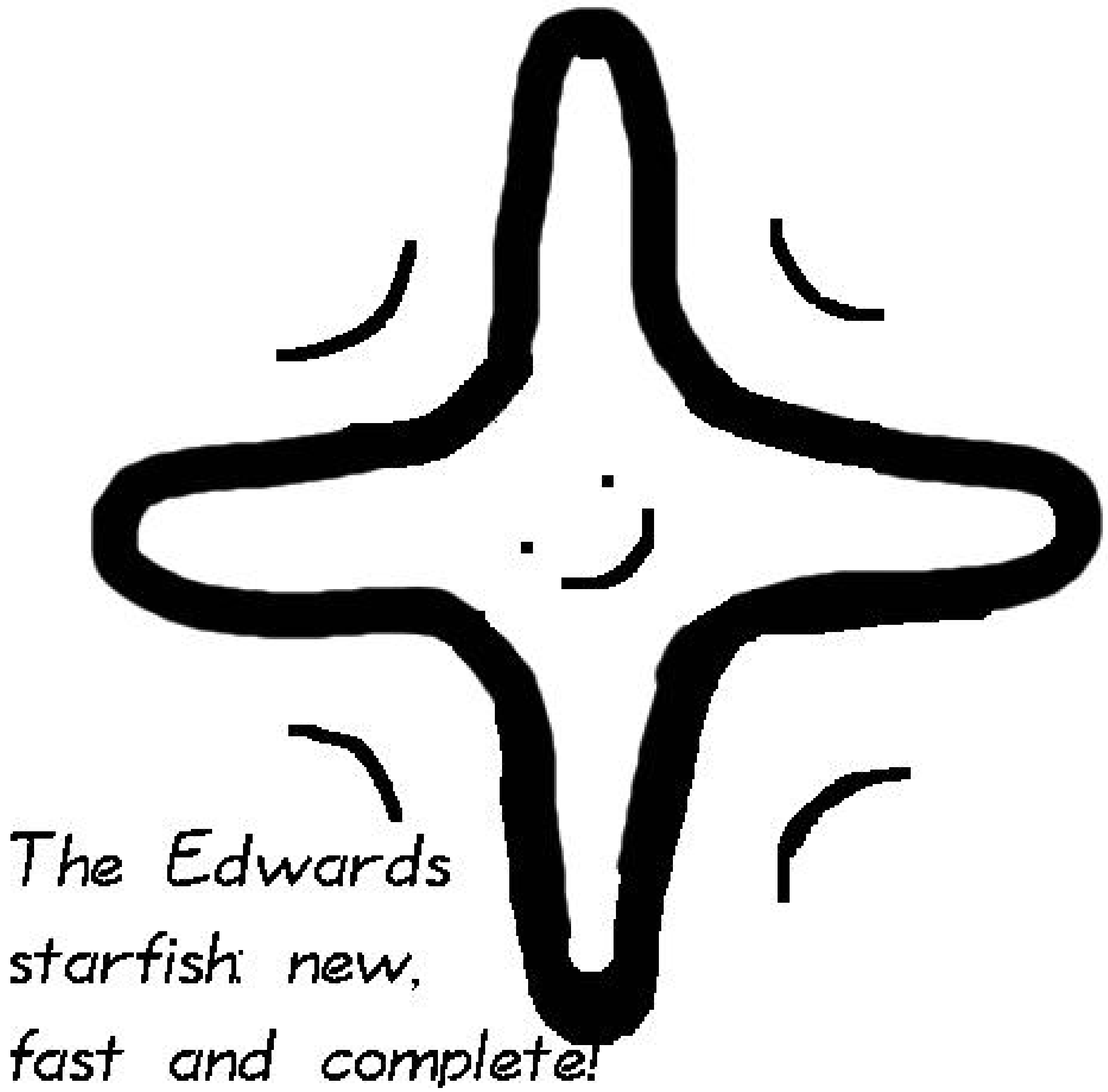


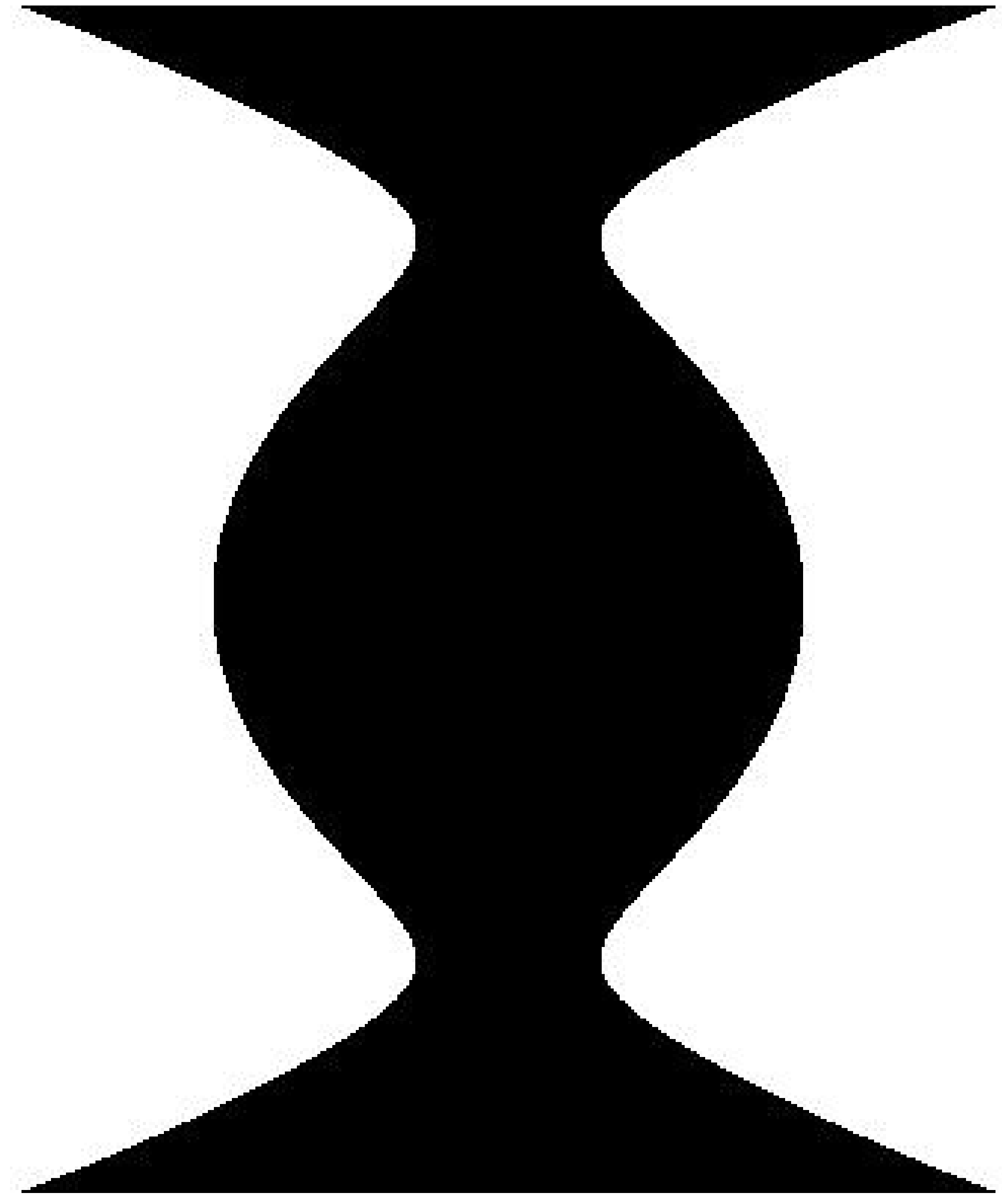
$x^2 = y^4 - 1.9y^2 +$



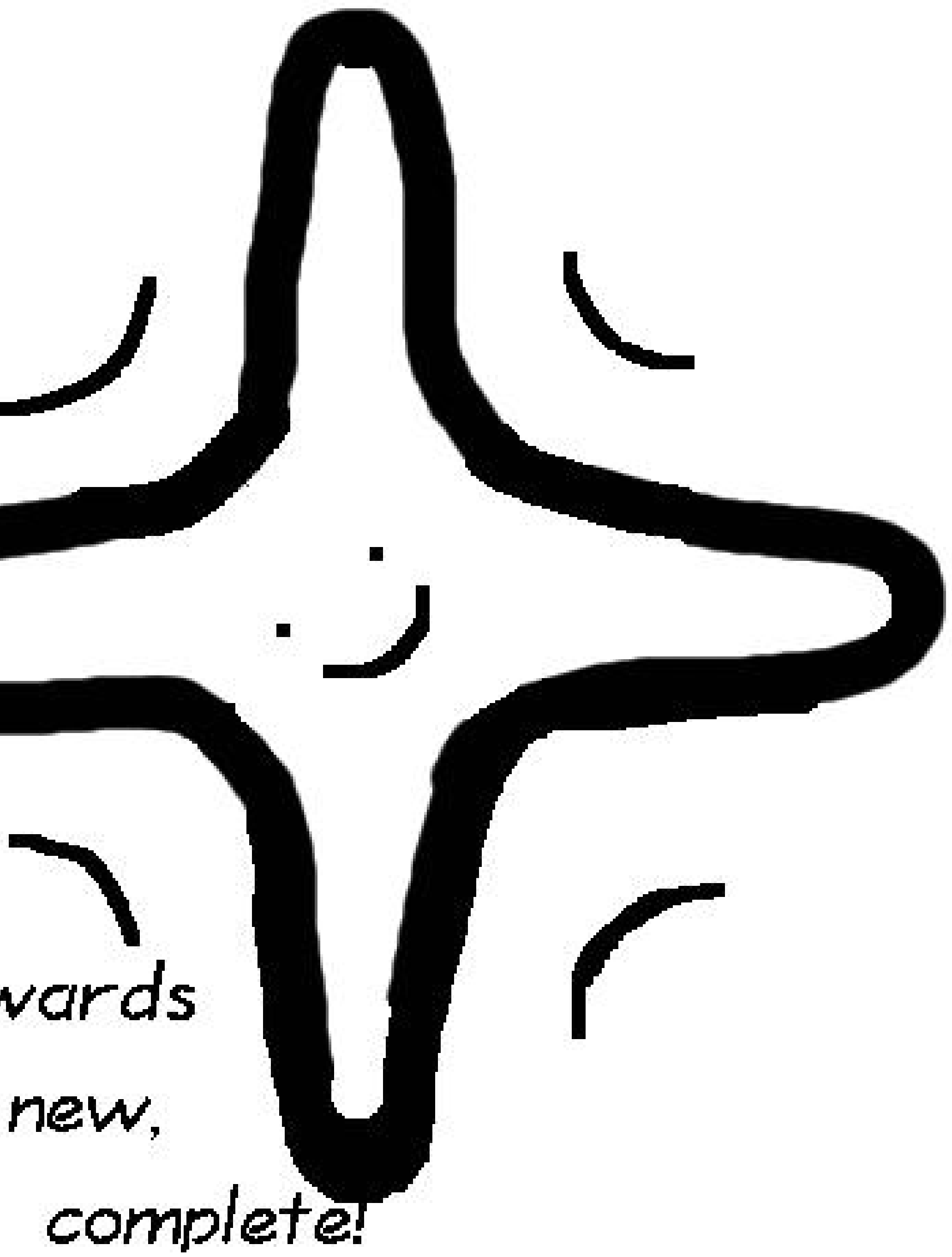


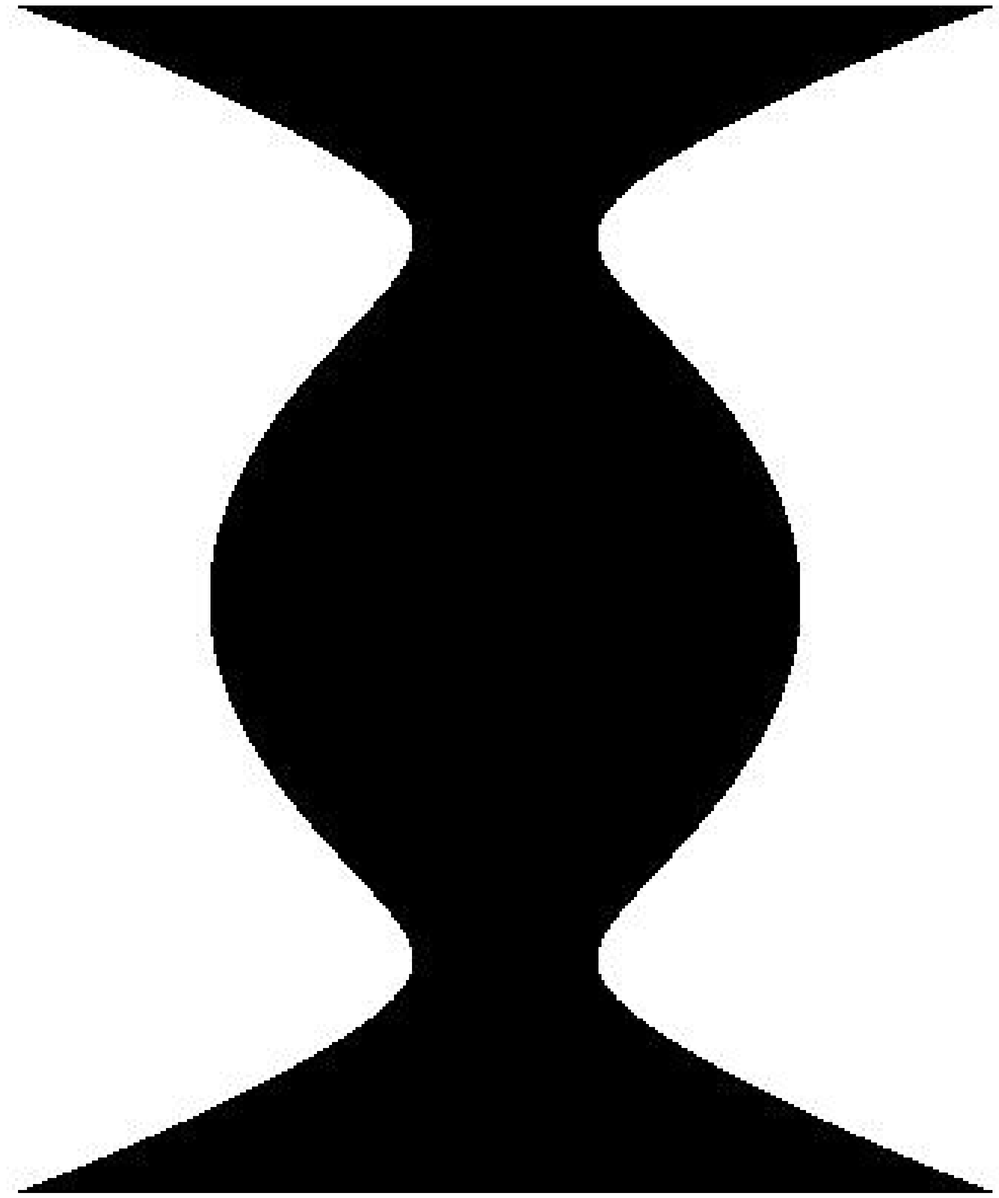
$$x^2 = y^4 - 1.9y^2 + 1$$





$x^2 = y^4 - 1.9y^2 + 1$





$$x^2 = y^4 - 1.9y^2 + 1$$

The Jac
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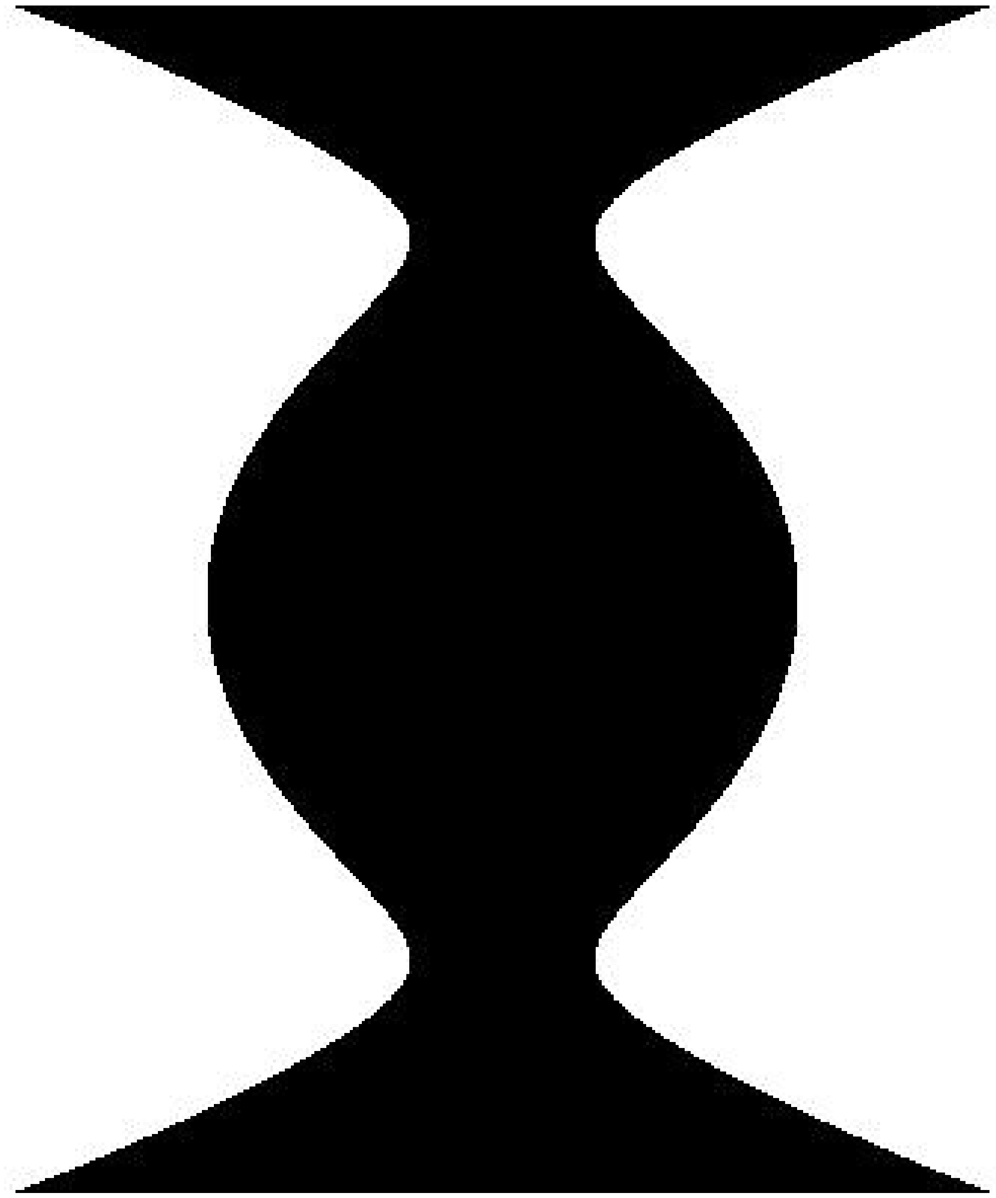
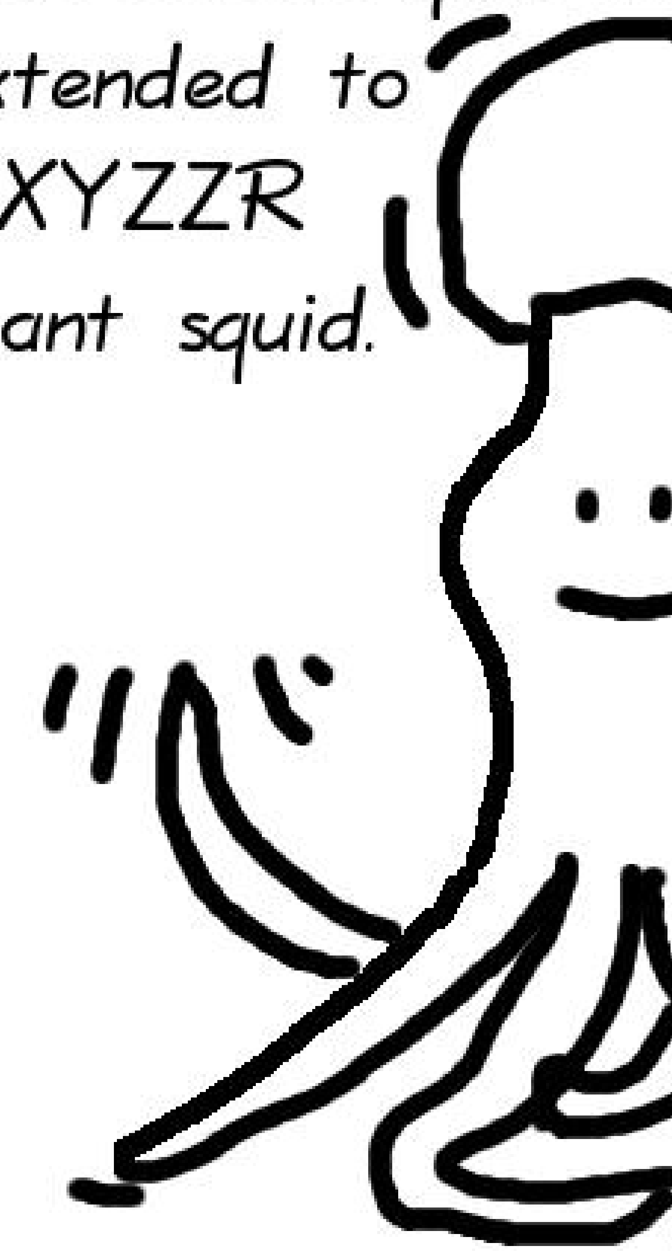
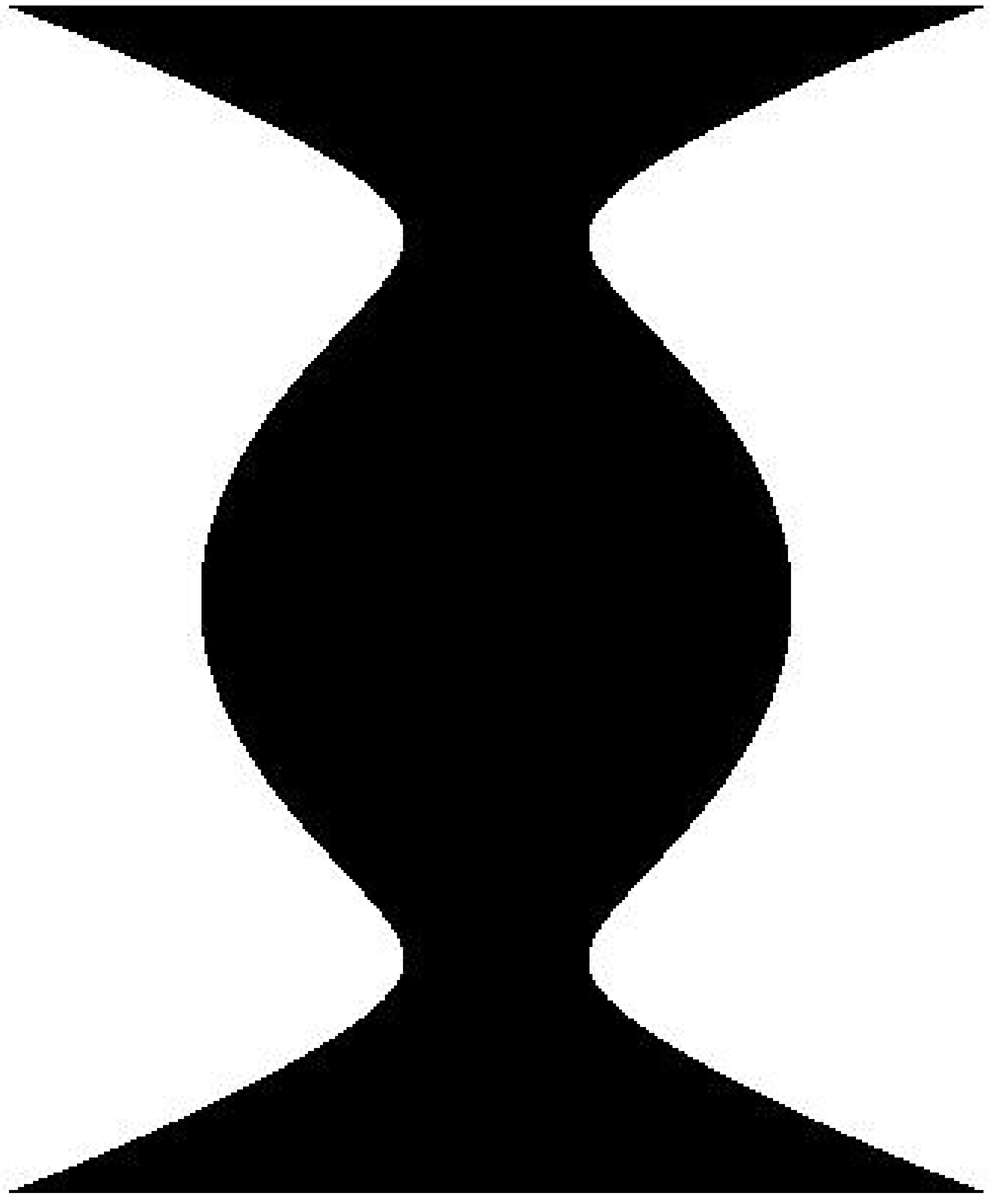
XXYZZ
giant sq



$$x^2 = y^4 - 1.9y^2 + 1$$

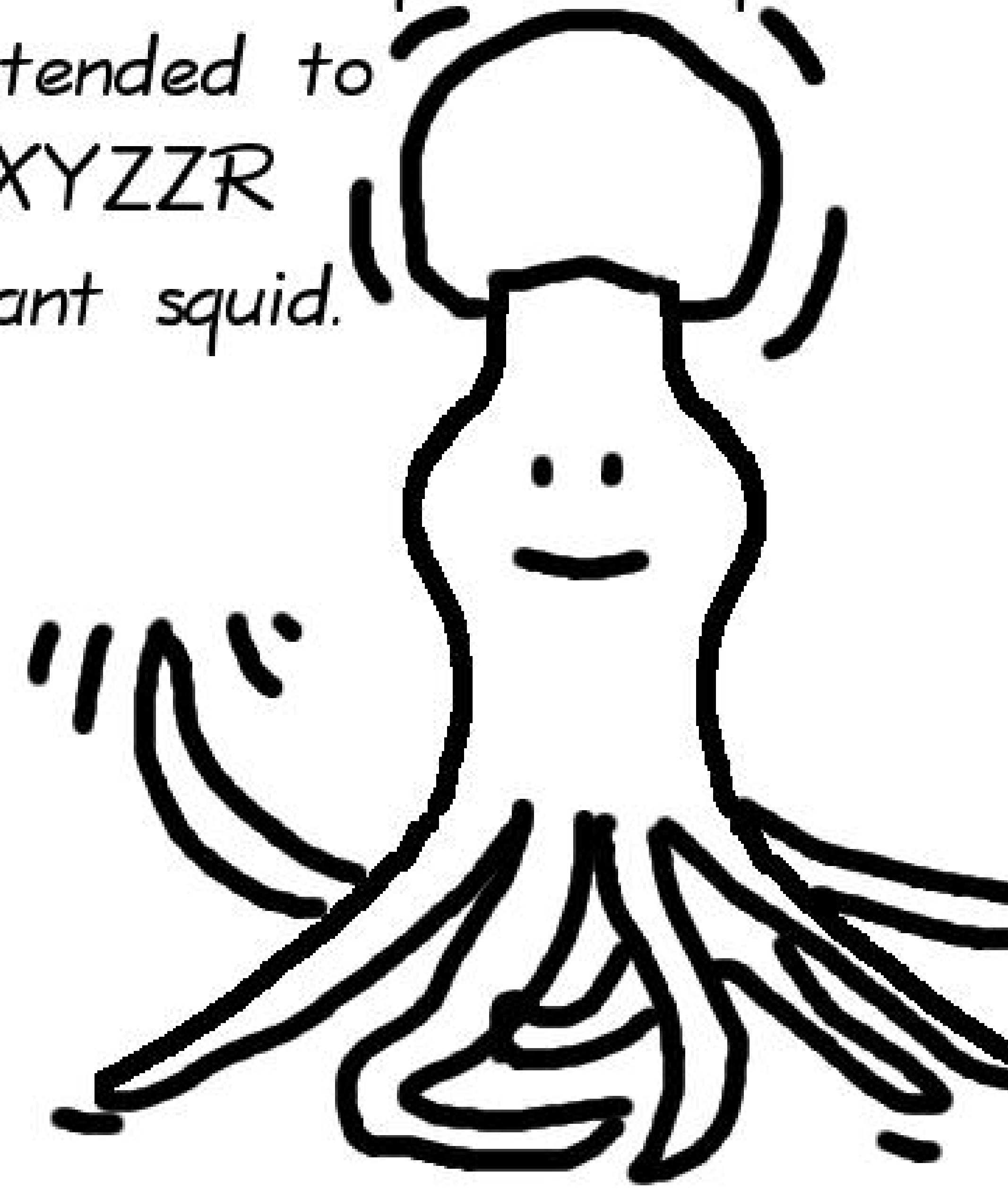
The Jacobi-quartic
extended to
XXYZZR
giant squid.

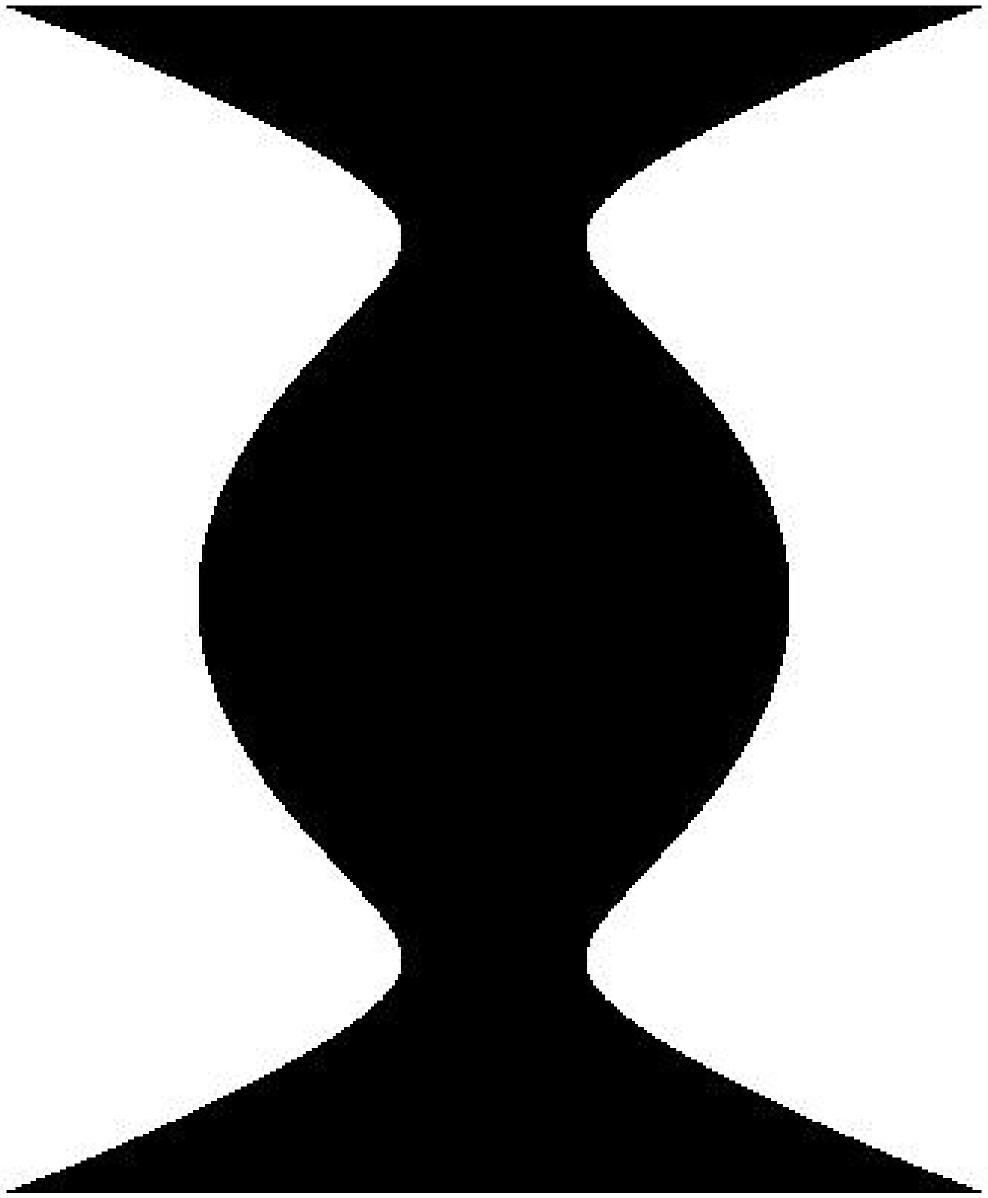




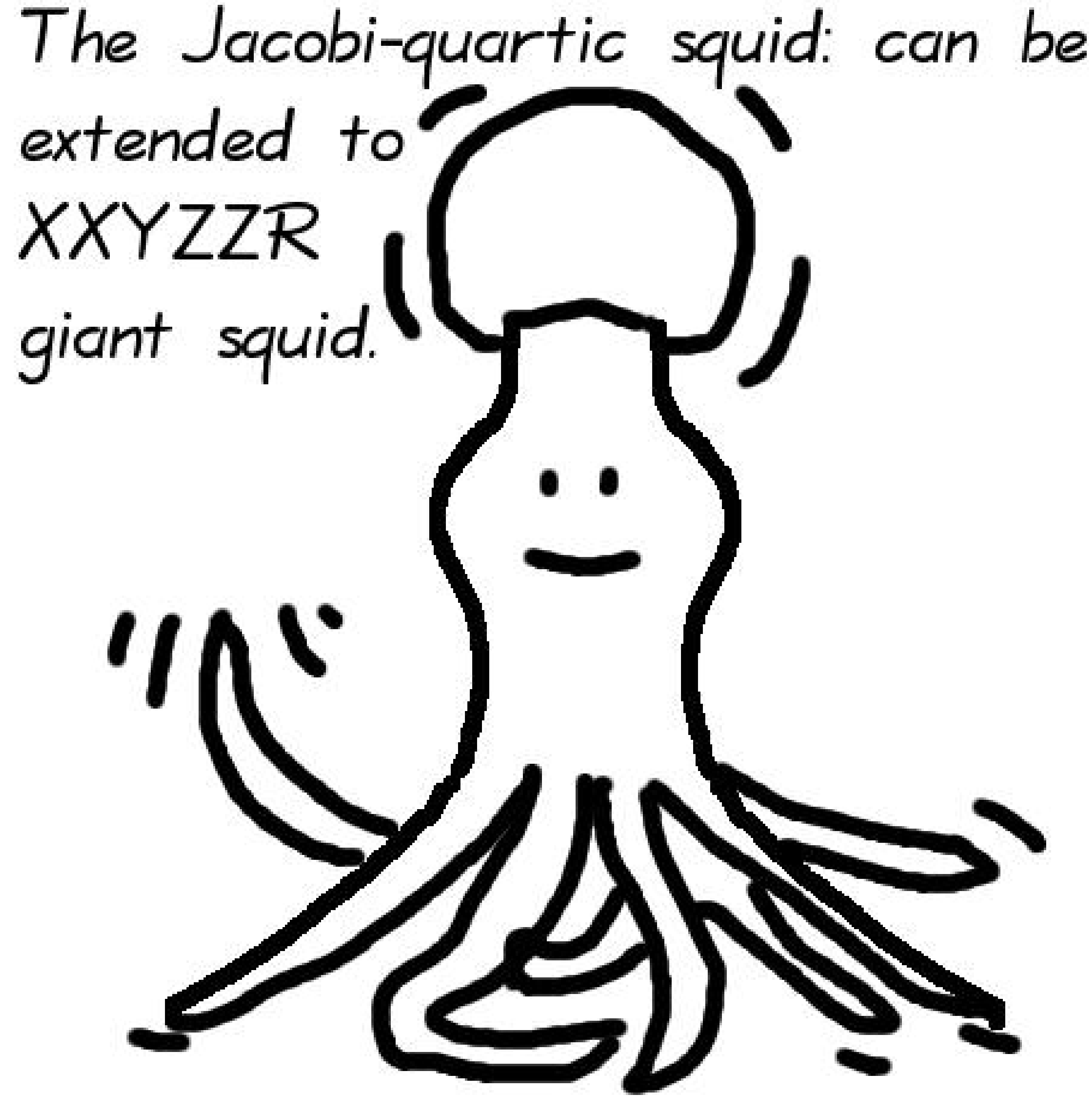
$$x^2 = y^4 - 1.9y^2 + 1$$

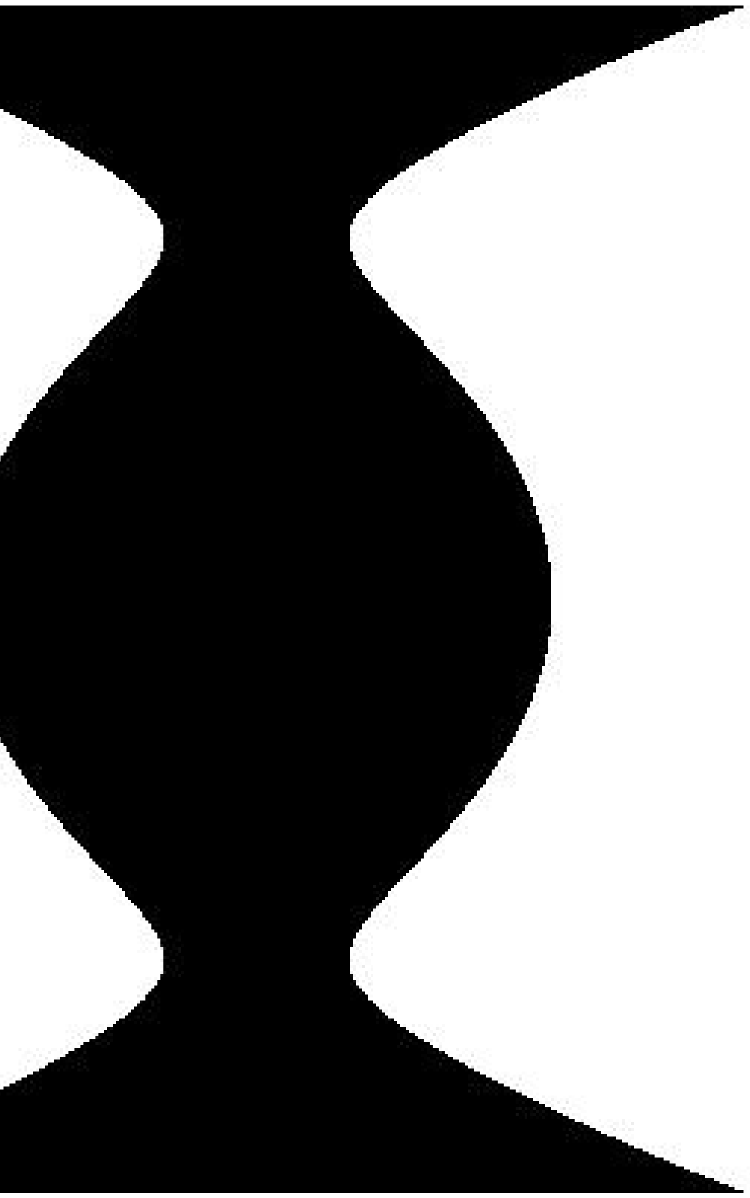
The Jacobi-quartic squid: ca
extended to
XXYZZR
giant squid.



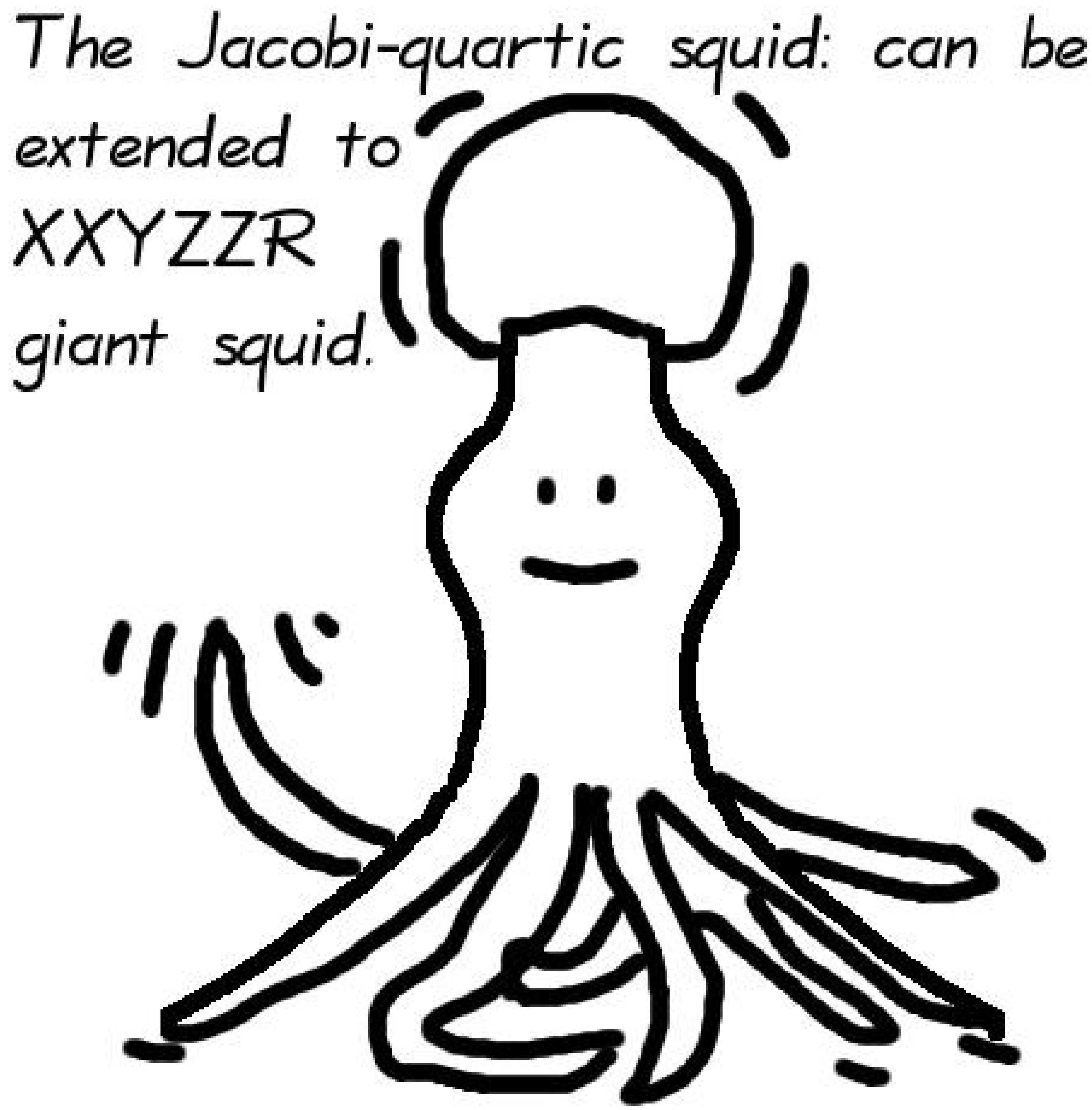


$x^2 = y^4 - 1.9y^2 + 1$

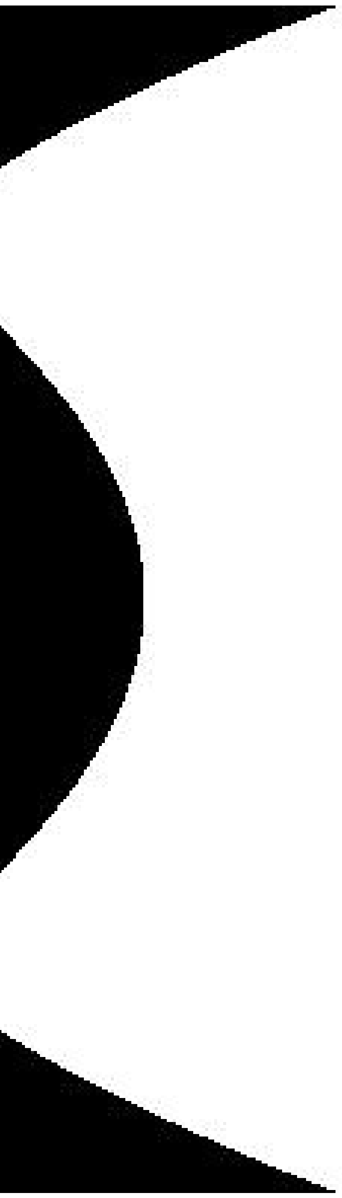

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



$- 1.9y^2 + 1$

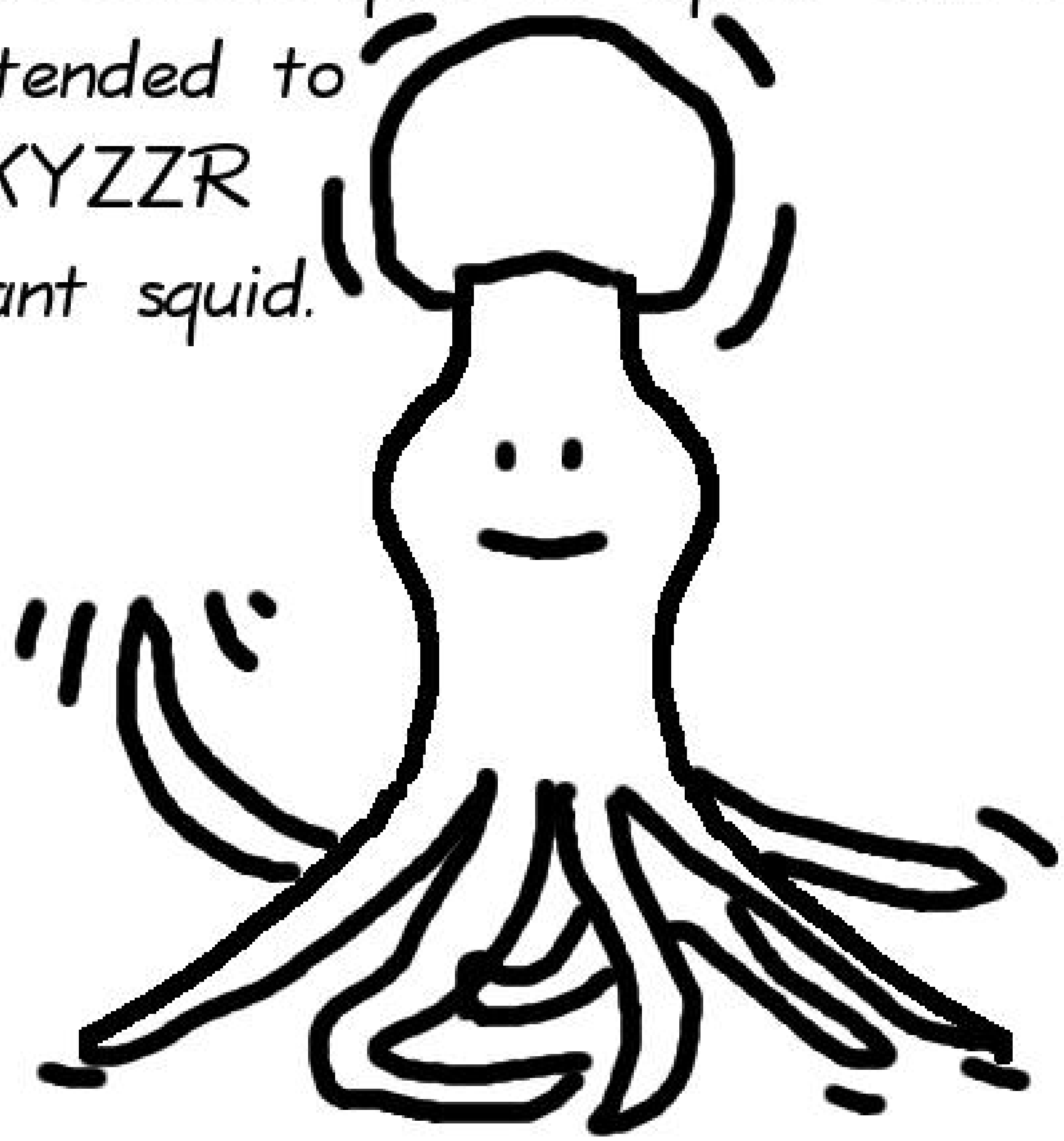

The Jacobi-quartic squid: can be
extended to
XXYZZR
giant squid.

$x^3 - y^3$



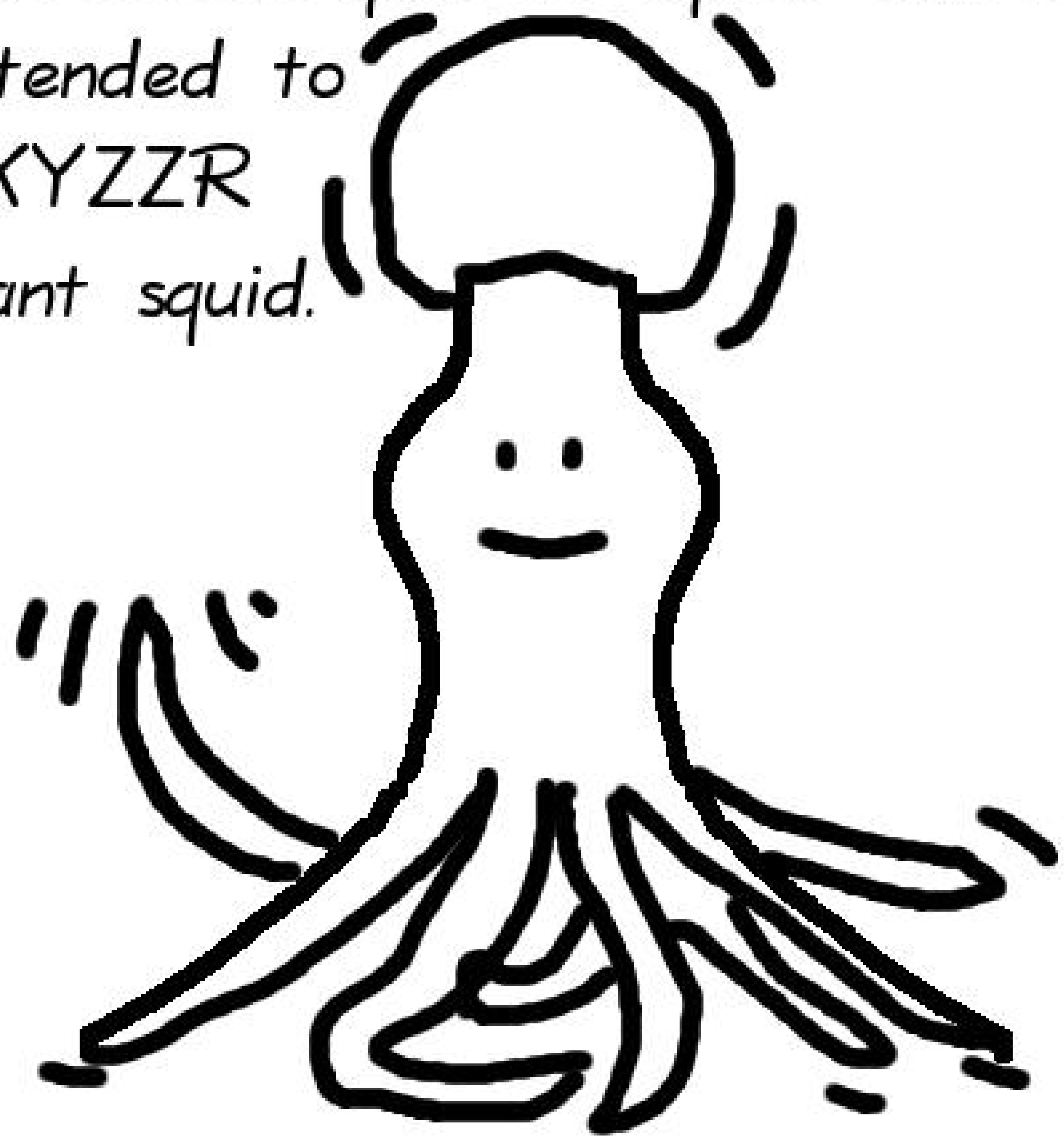
$- 1$

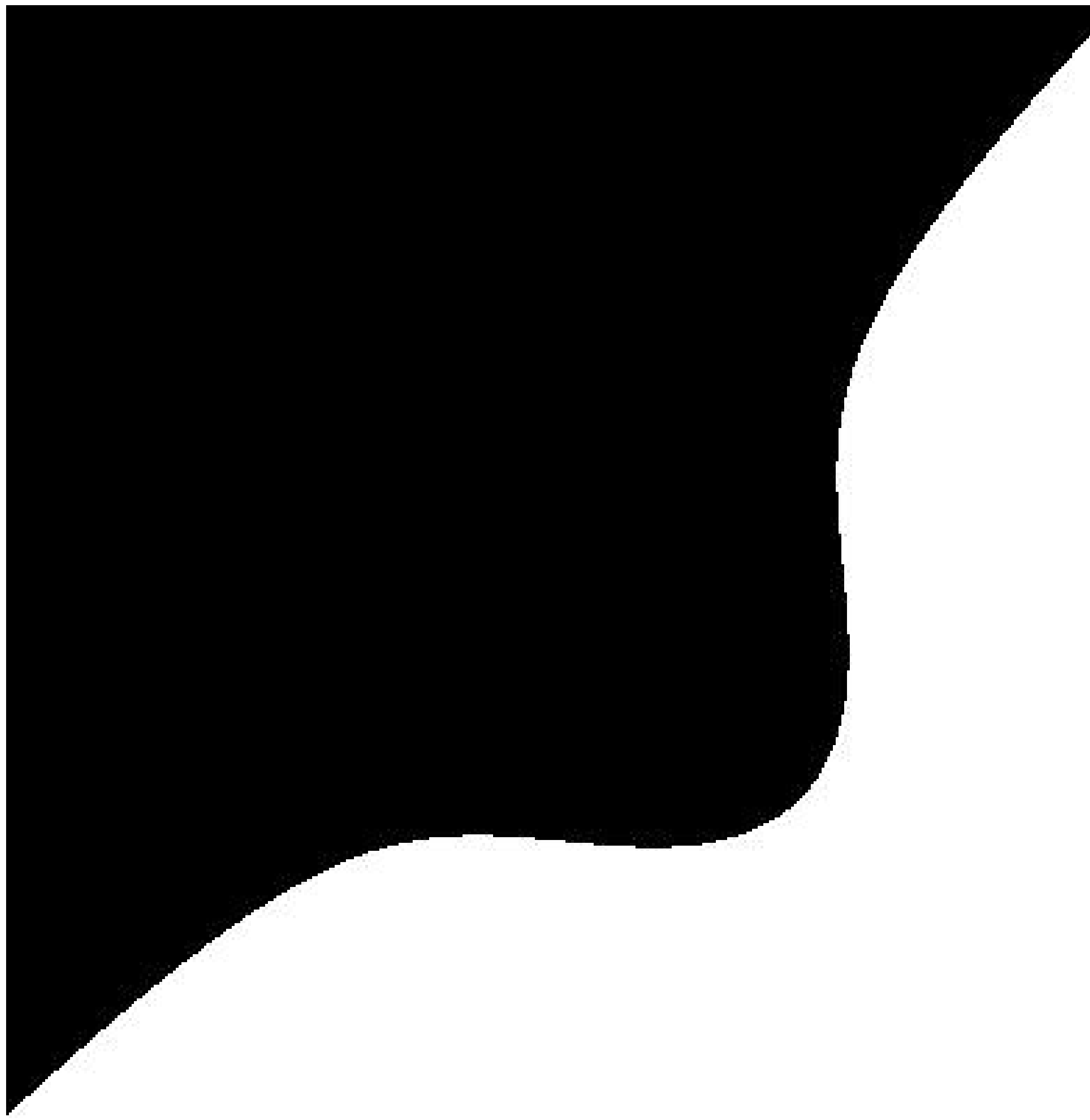
The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



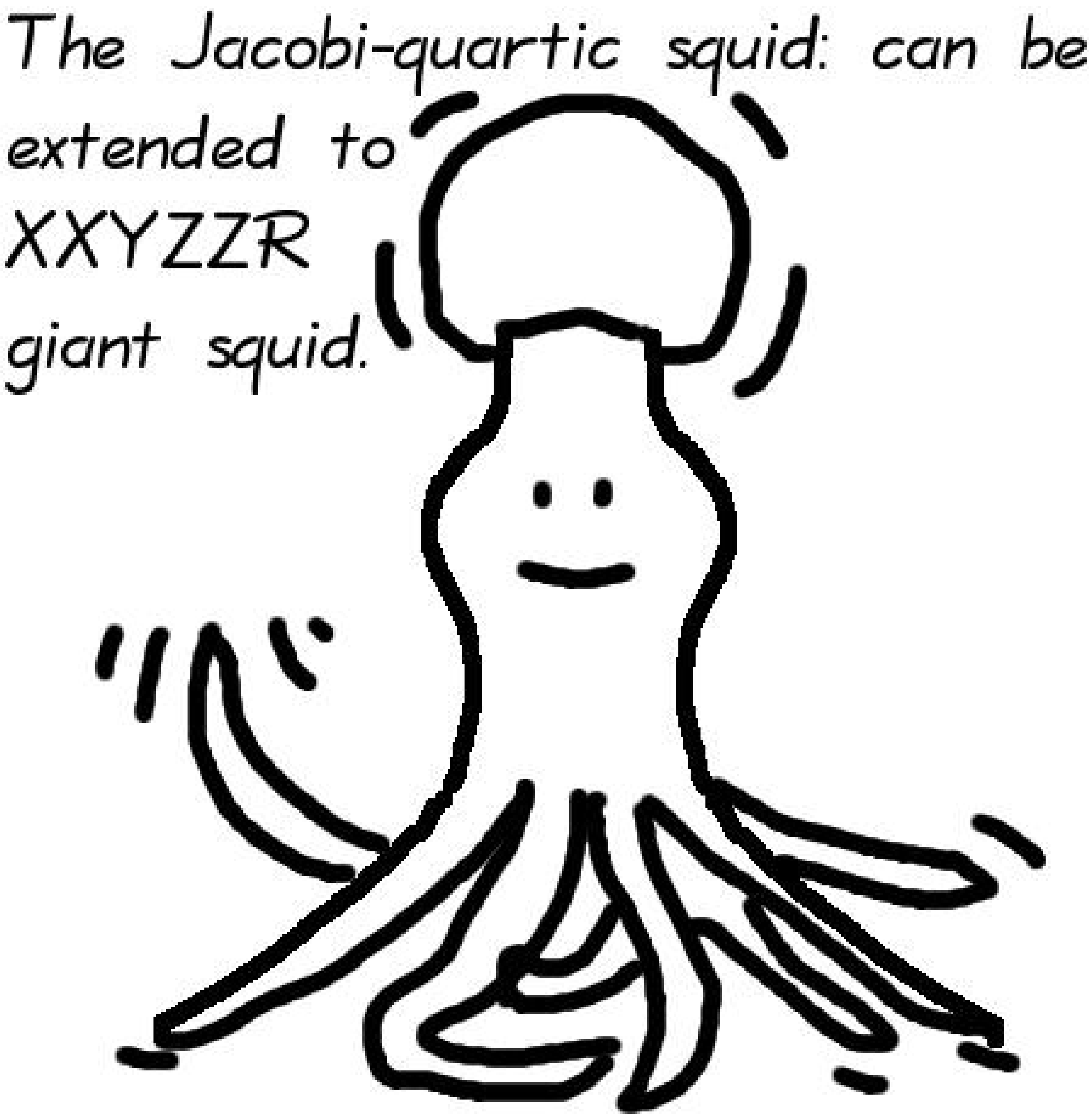
$x^3 - y^3 + 1 = 0.3$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

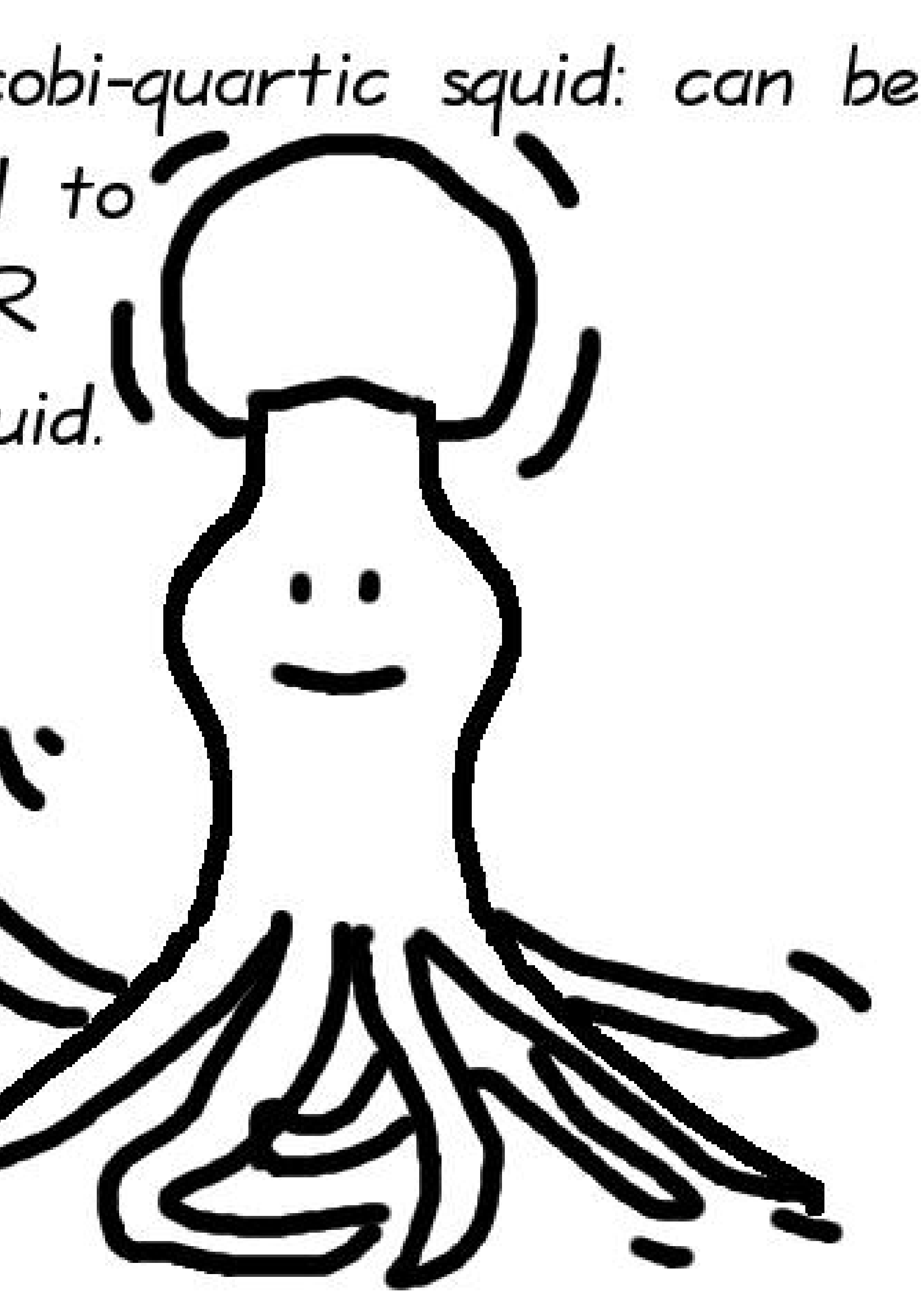




$$x^3 - y^3 + 1 = 0.3xy$$

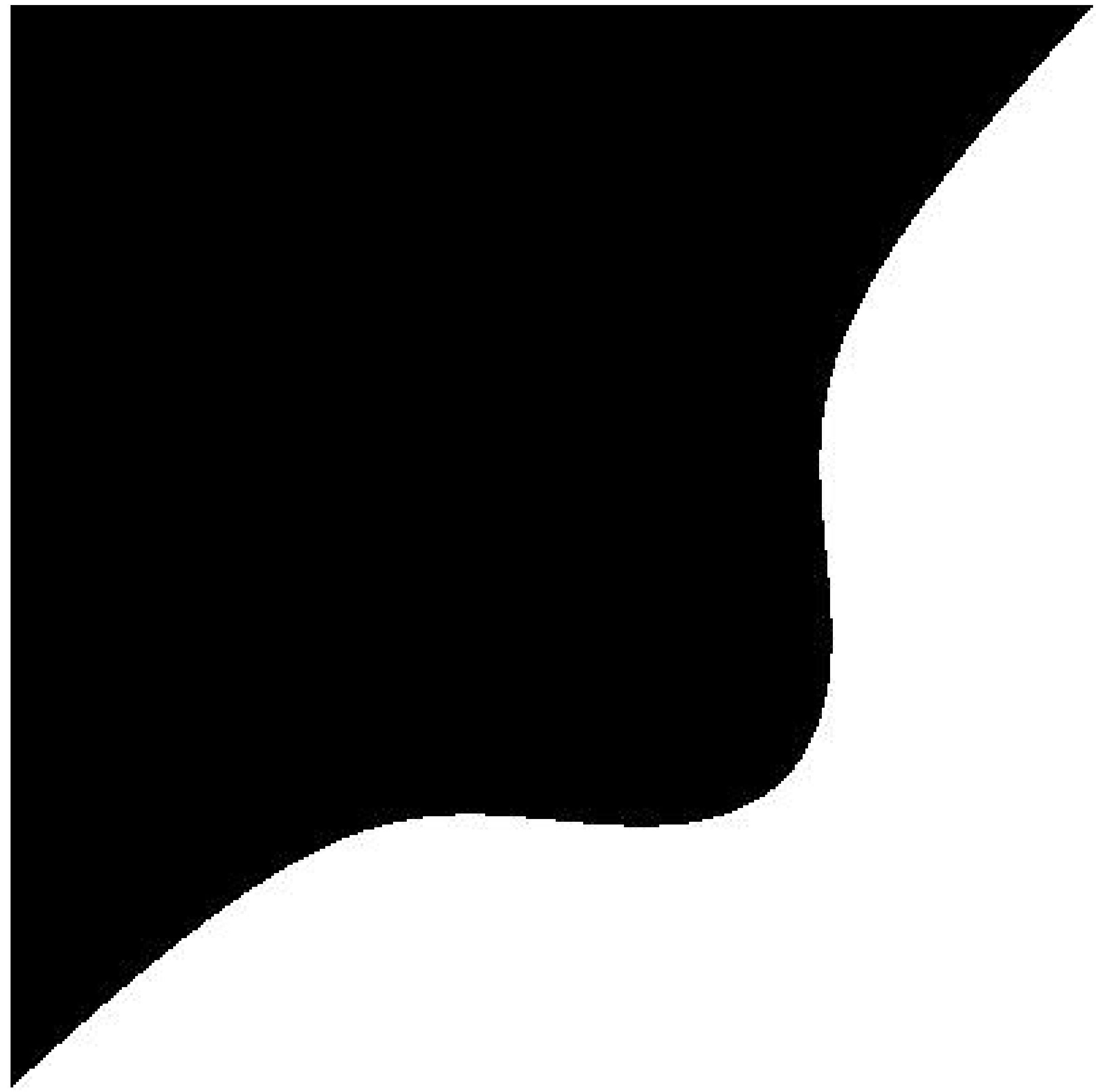

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

$$x^3 - y^3 + 1 = 0.3xy$$



$$x^3 - y^3 + 1 = 0.3xy$$

squid: can be



$$x^3 - y^3 + 1 = 0.3xy$$

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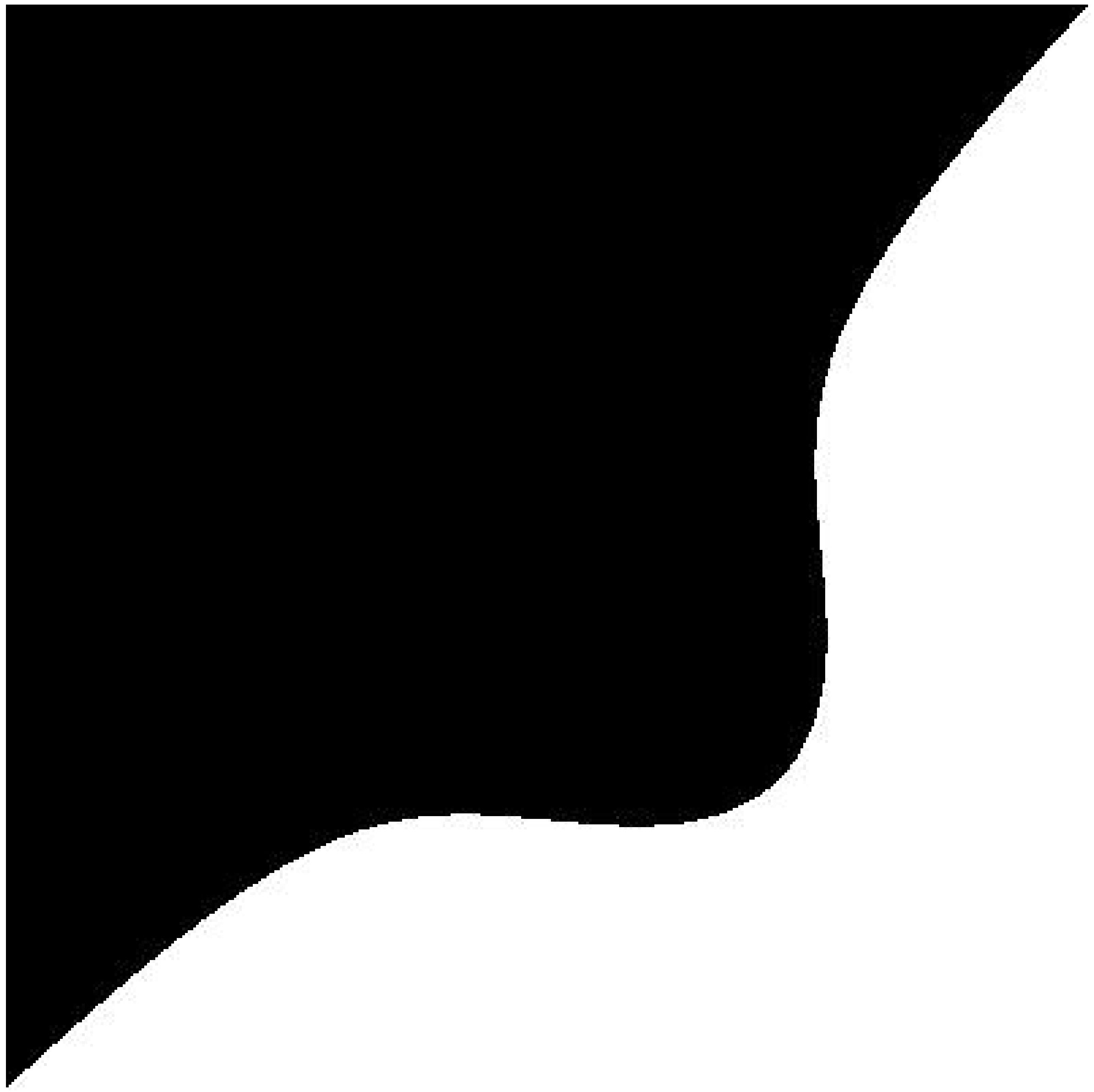
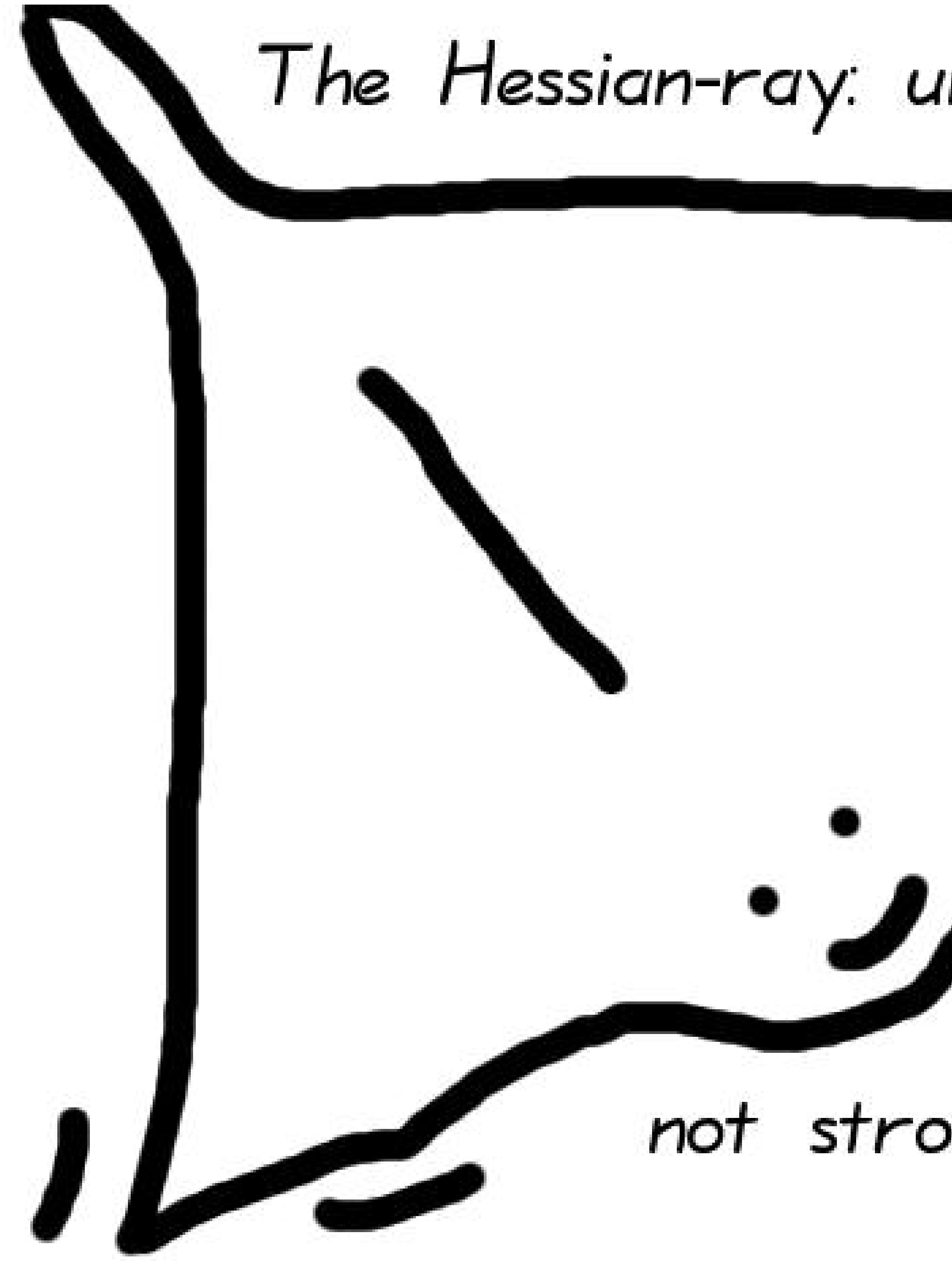
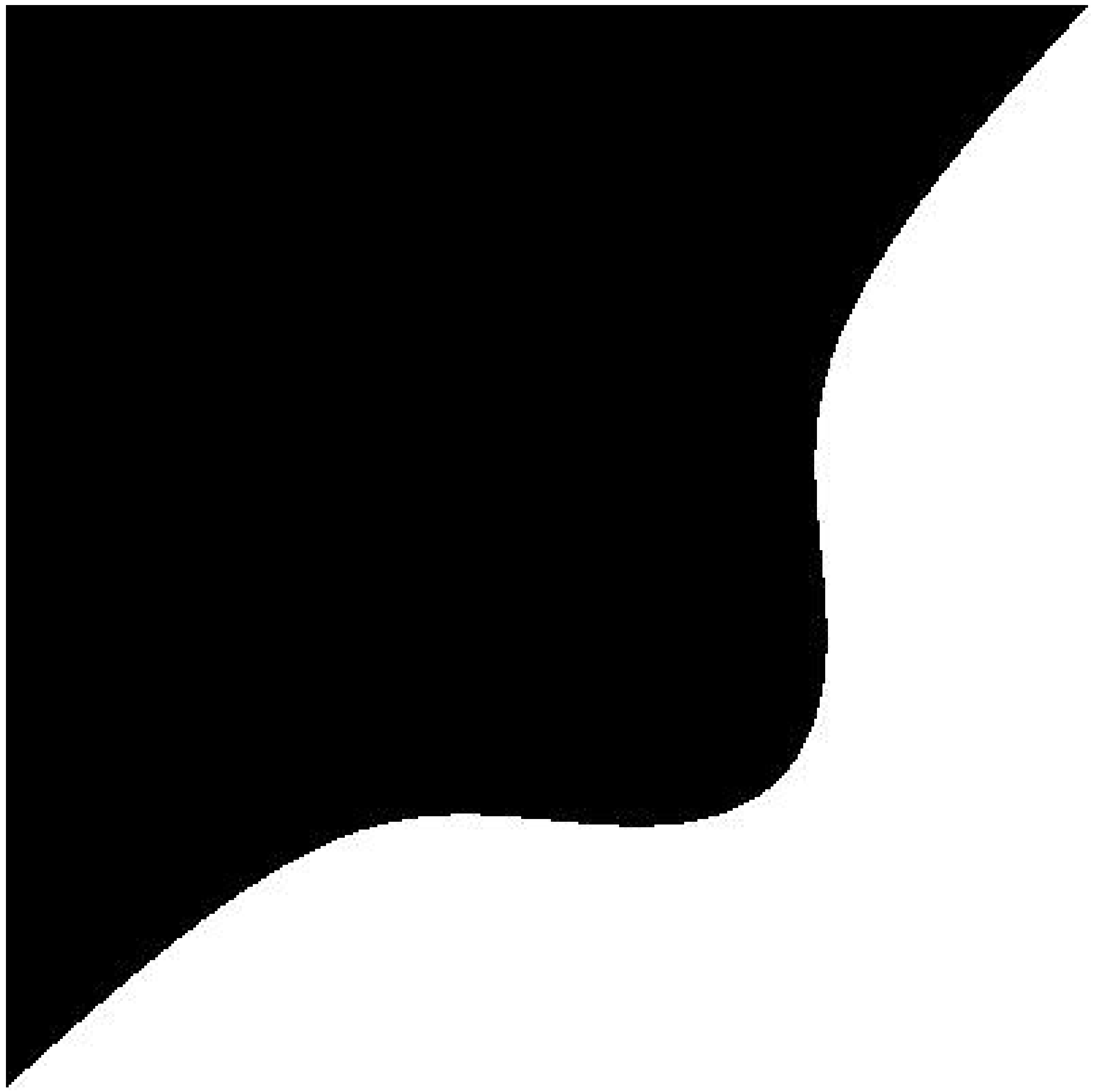
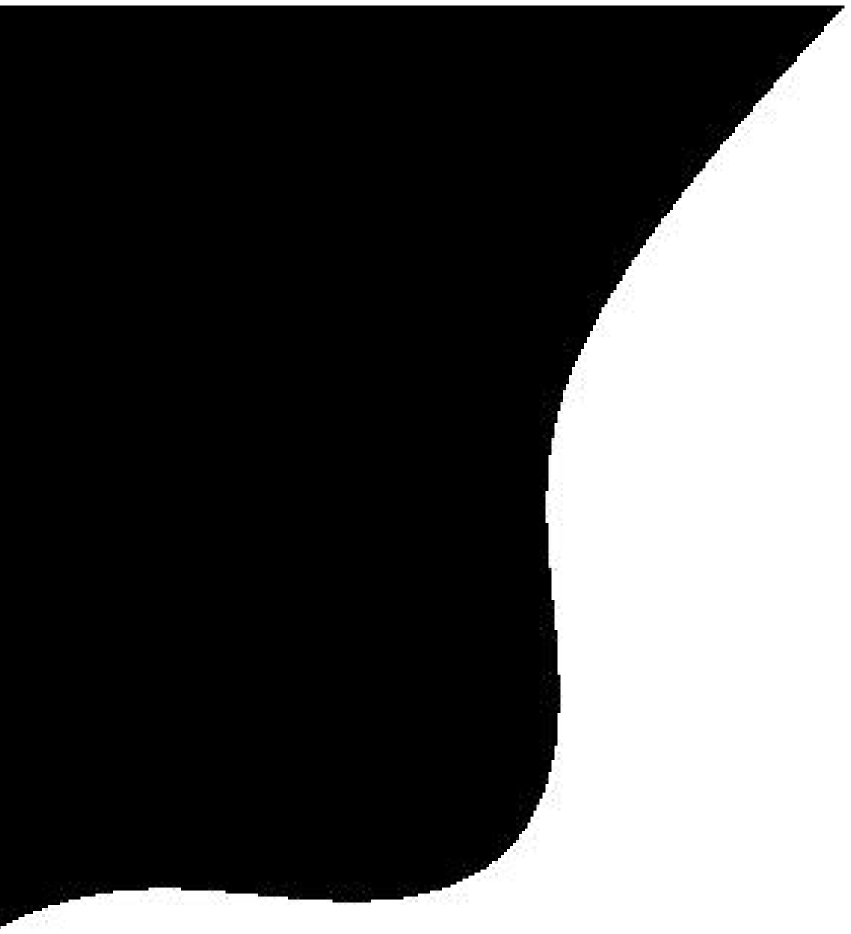


$x^3 - y^3 + 1 = 0.3xy$





$$x^3 - y^3 + 1 = 0.3xy$$



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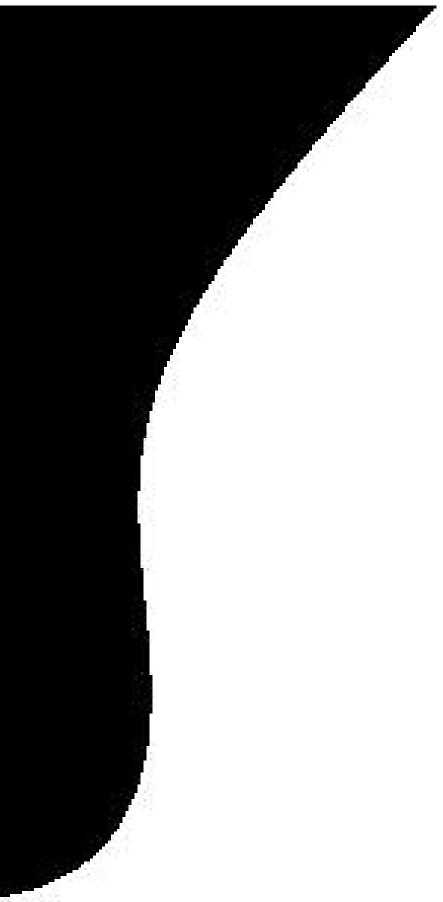

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The Hessian-ray: uniform
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not strongly so


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The Hessian-ray: uniform
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The Hessian-ray: uniform
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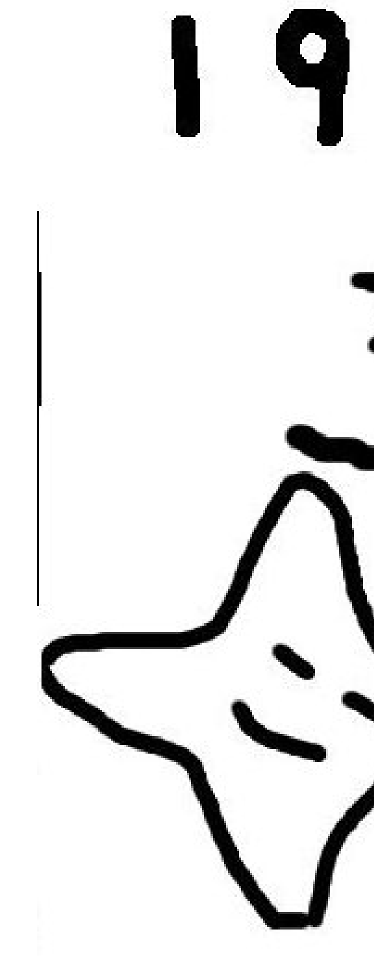
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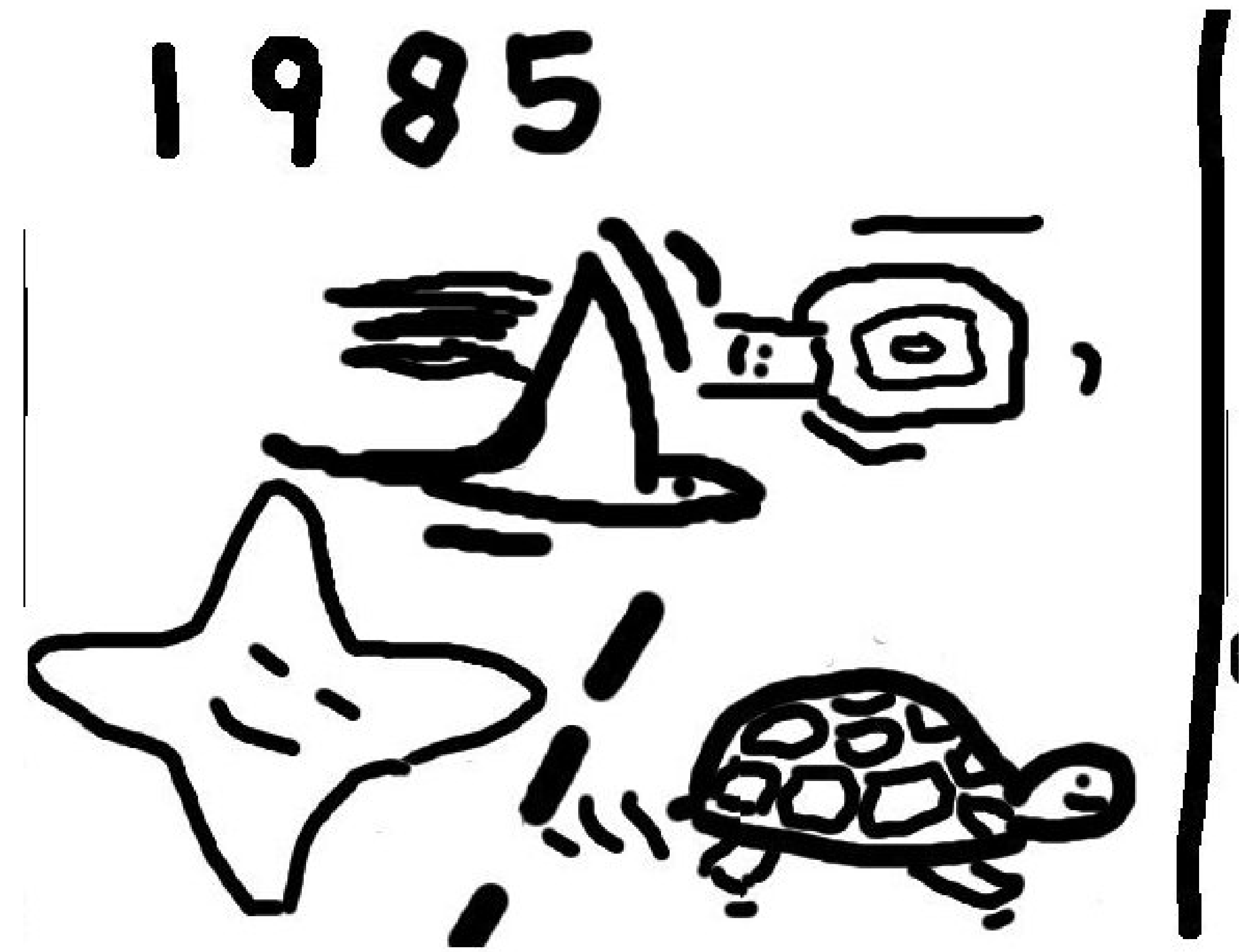
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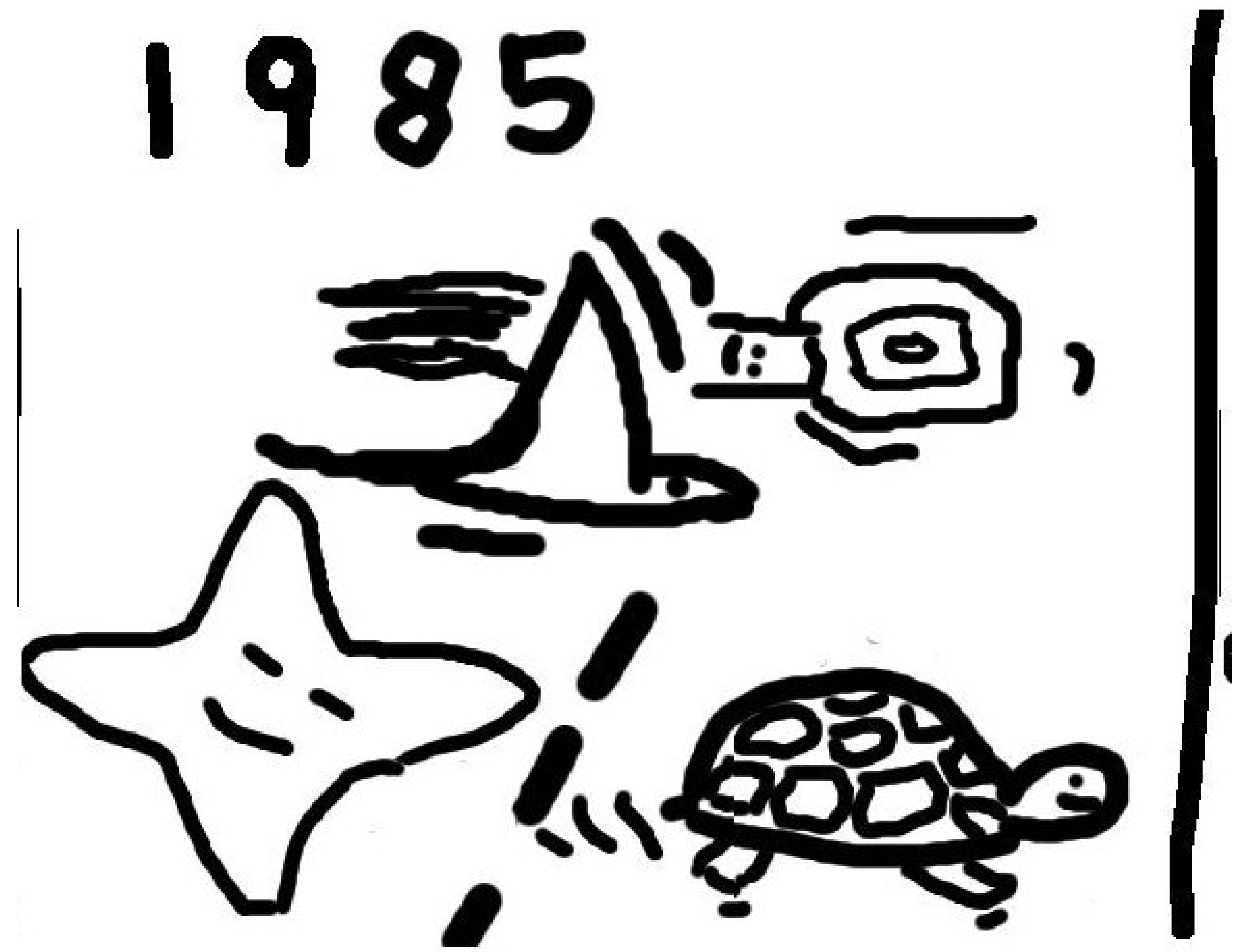
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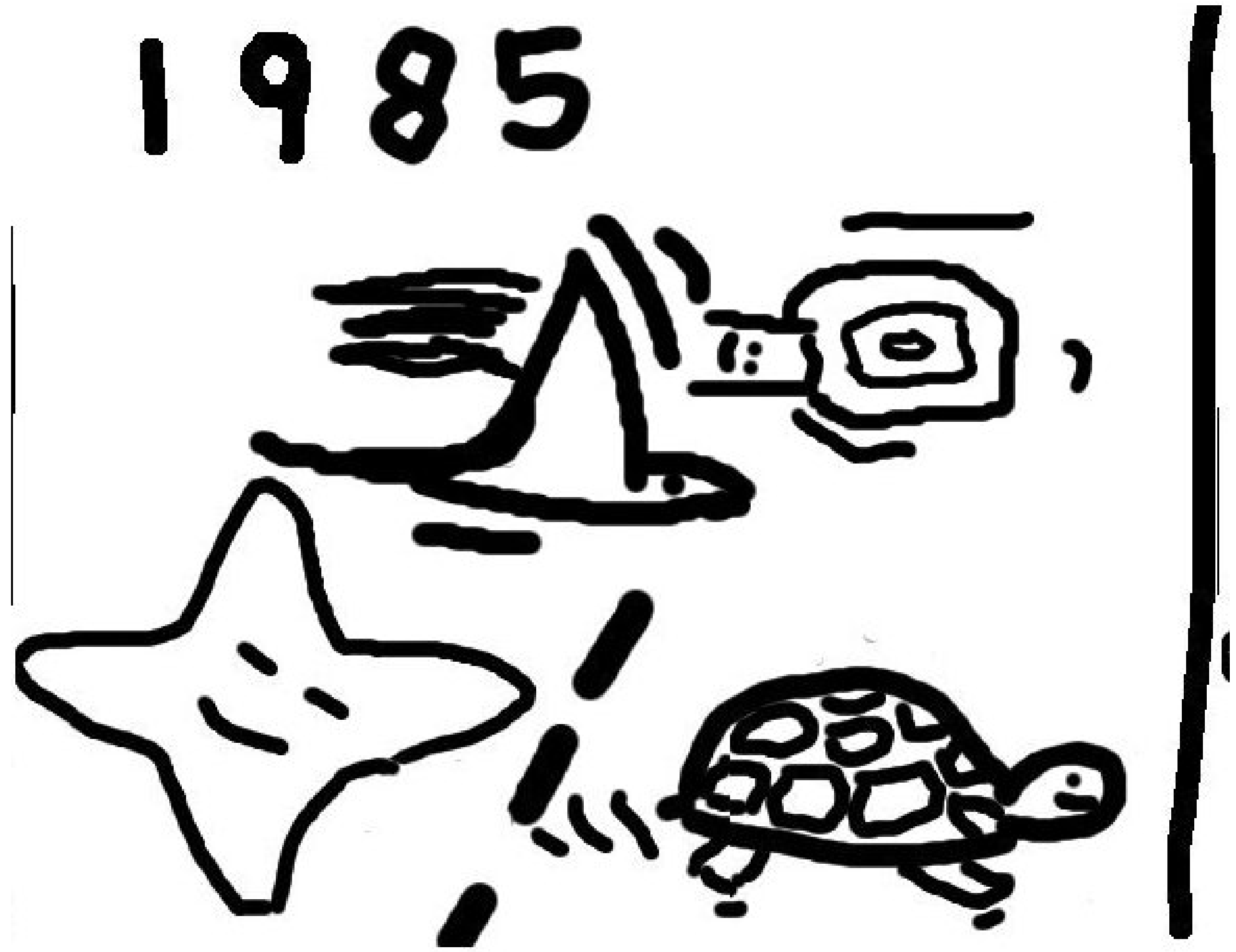
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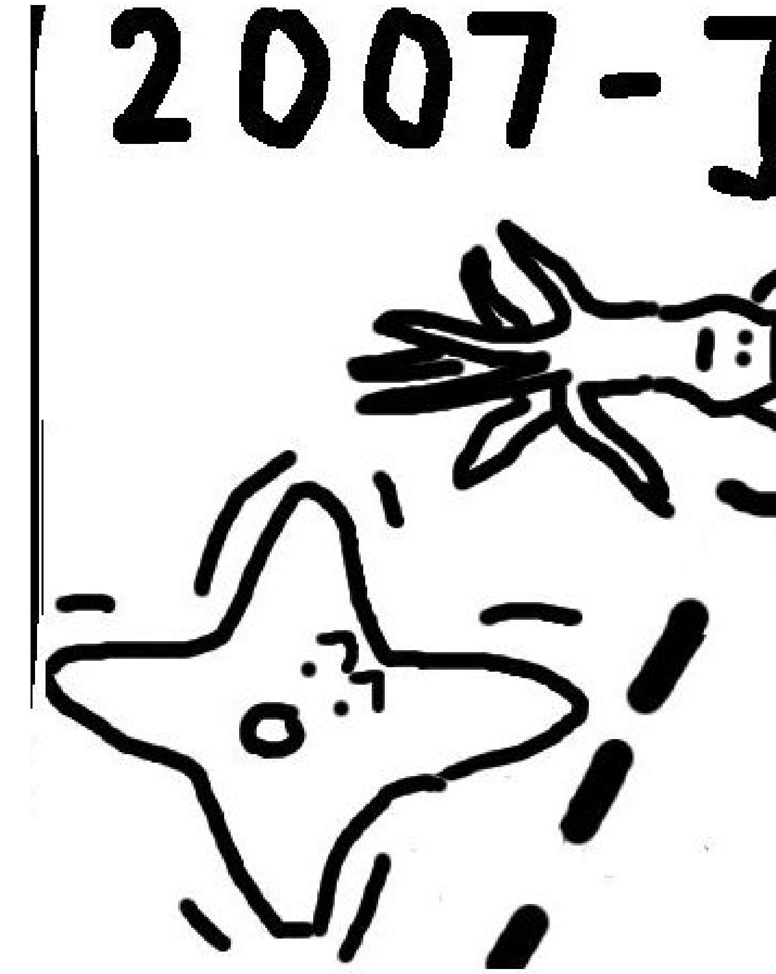
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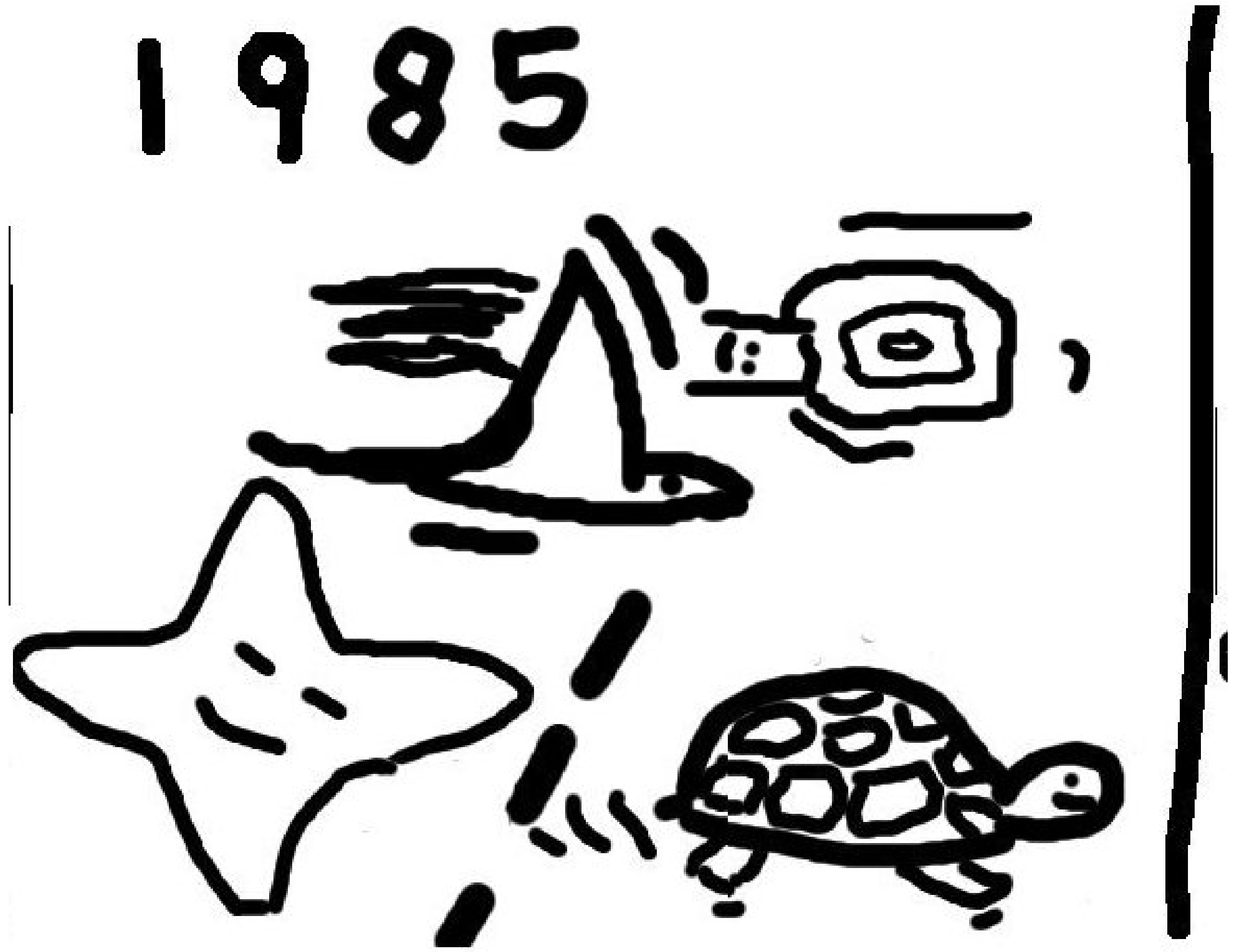
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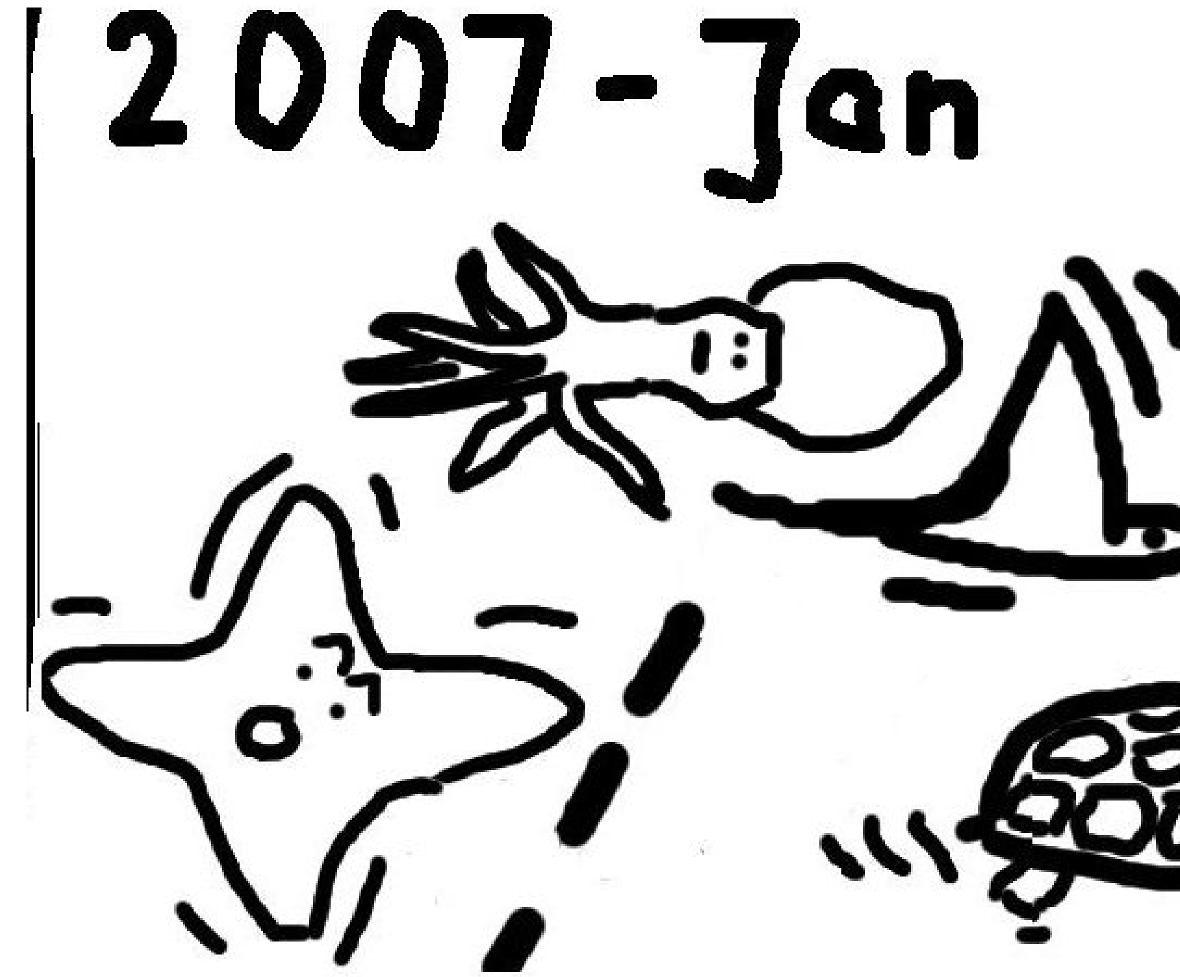
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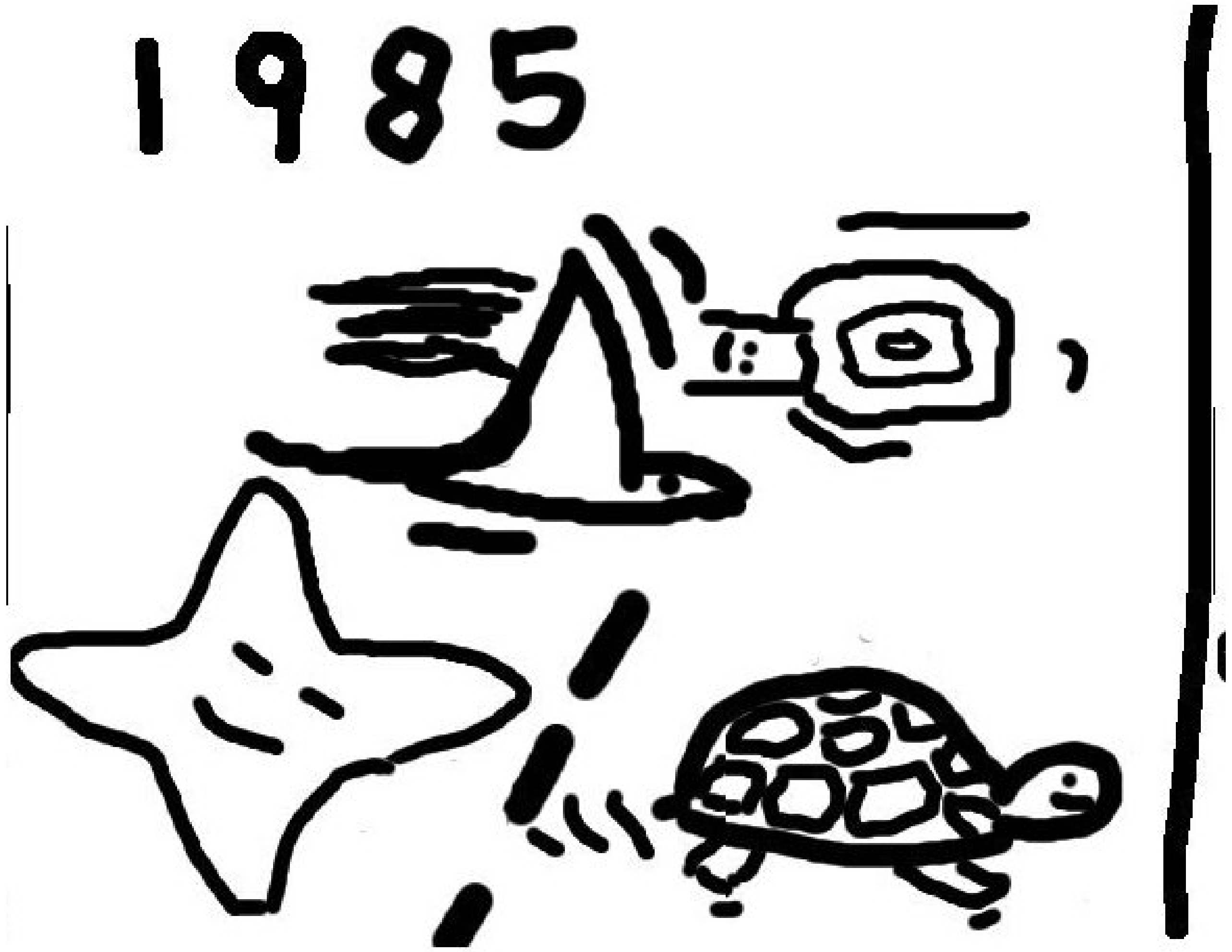
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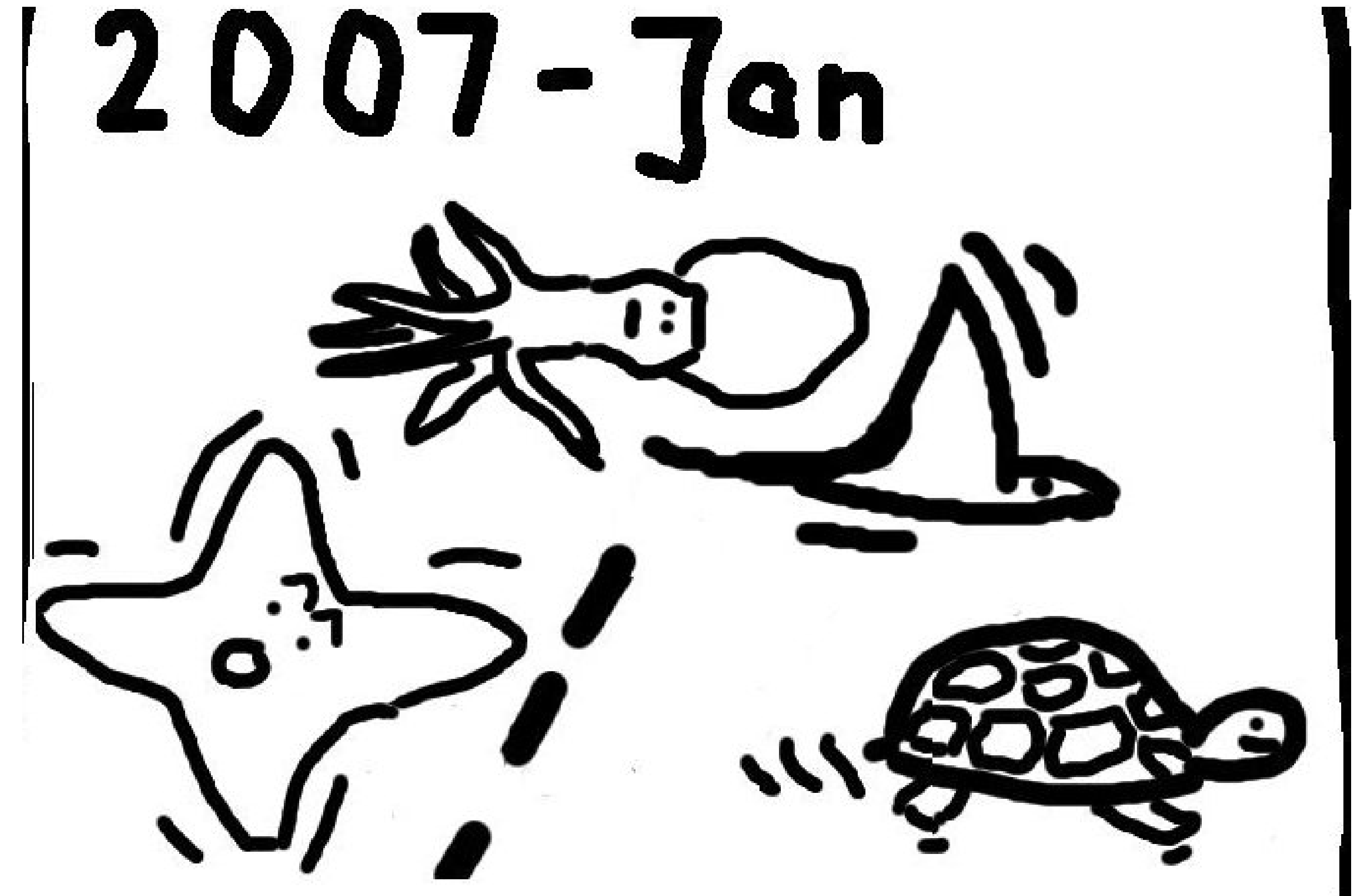
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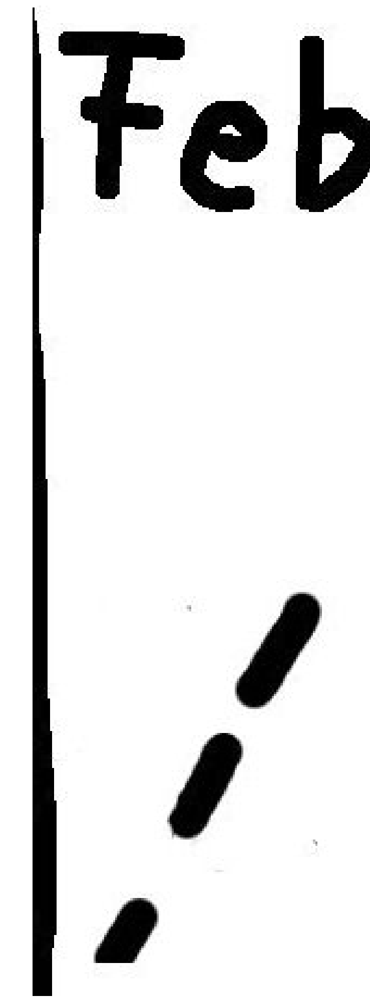
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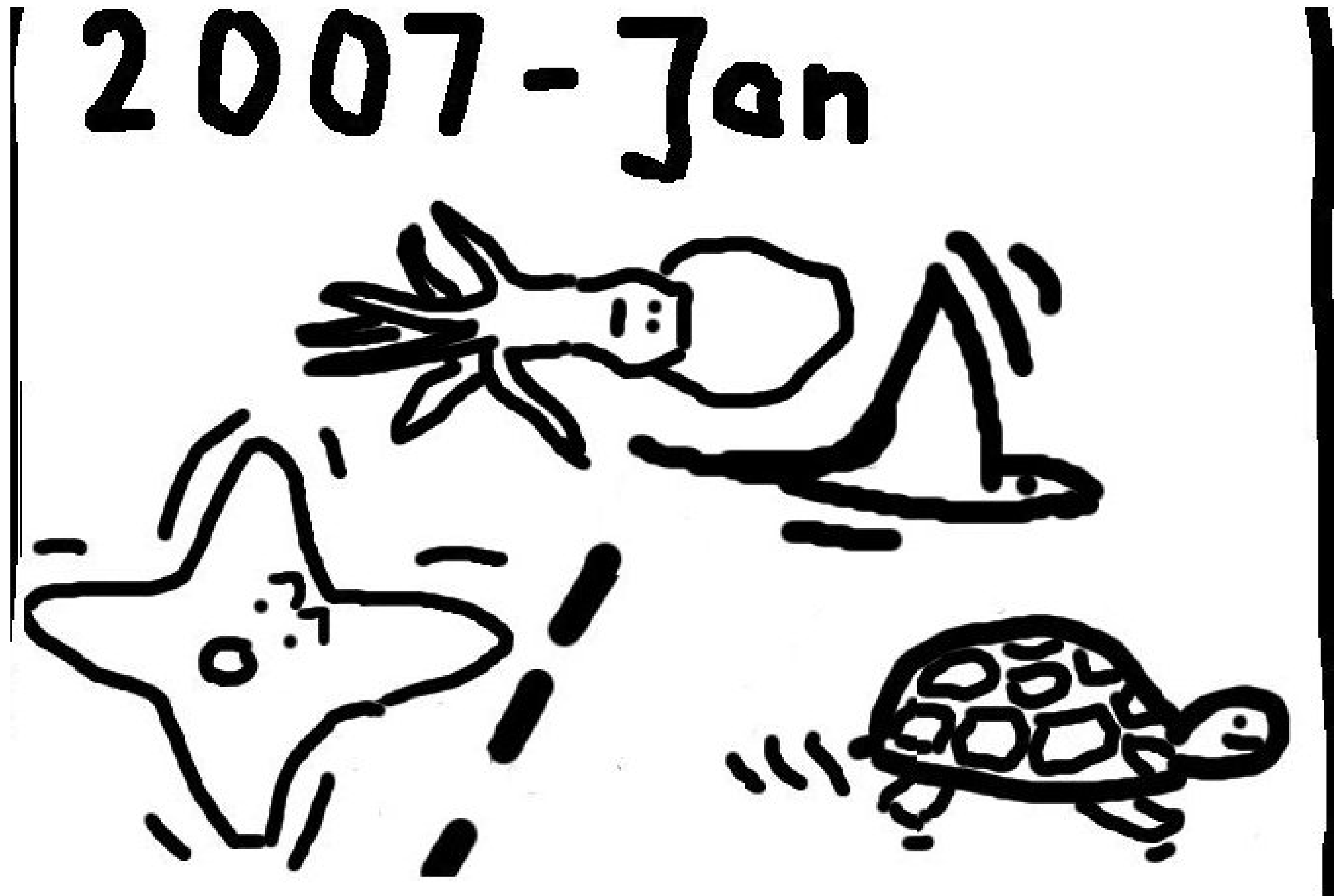
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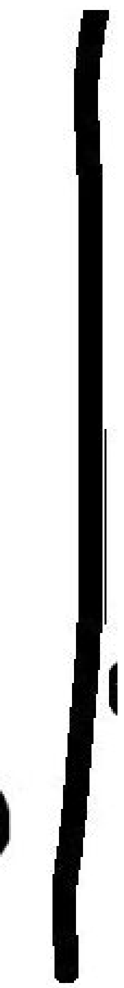

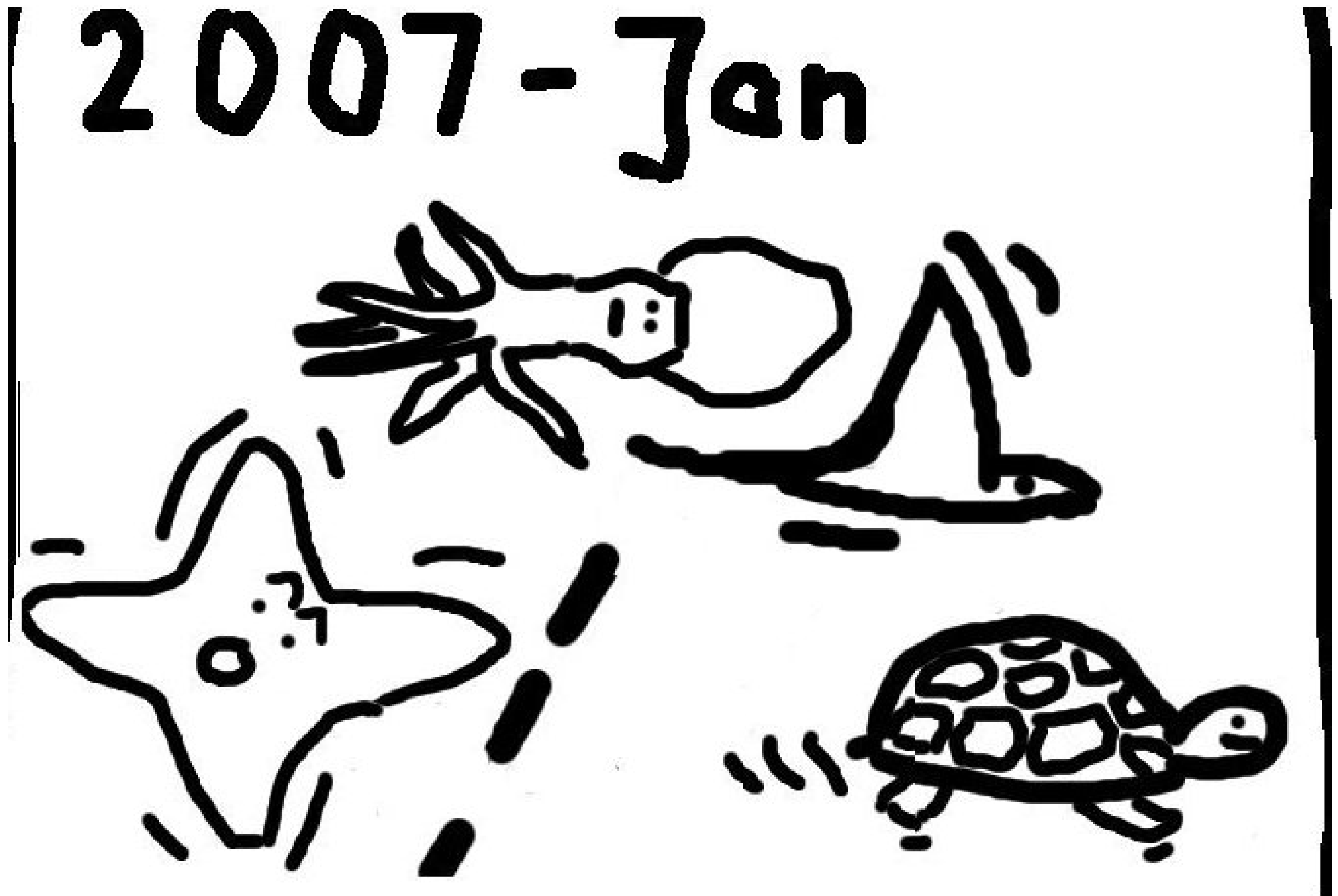
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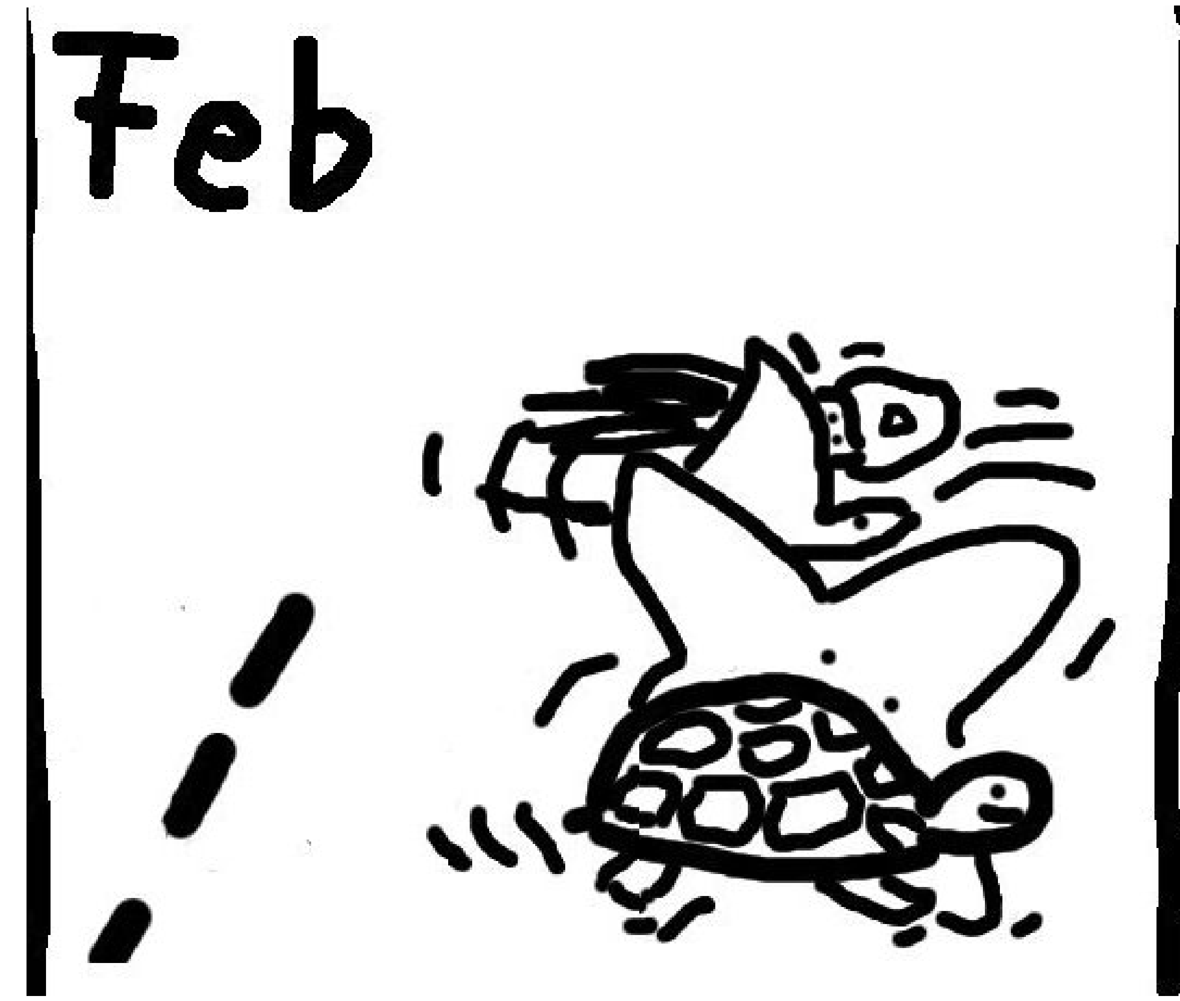
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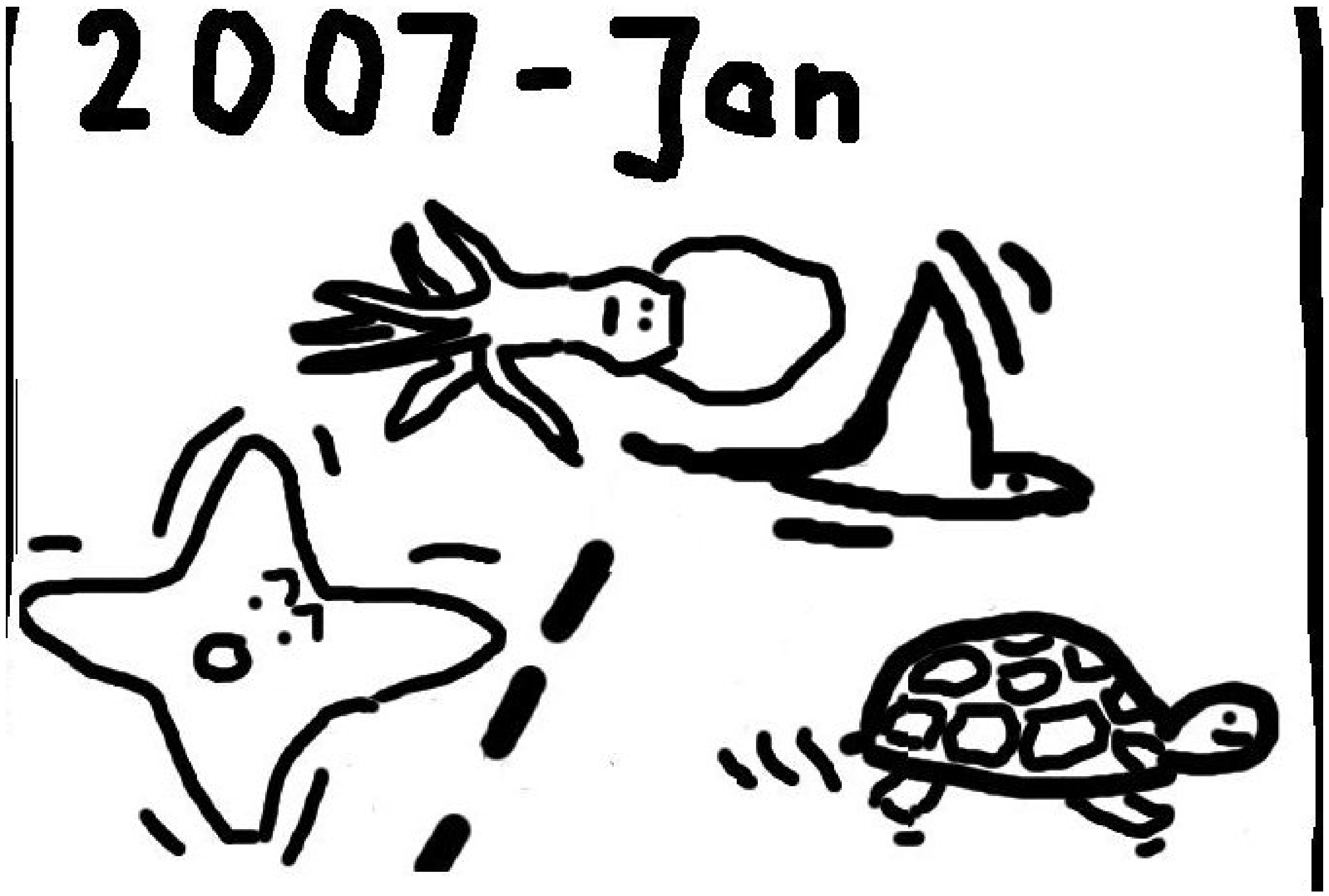
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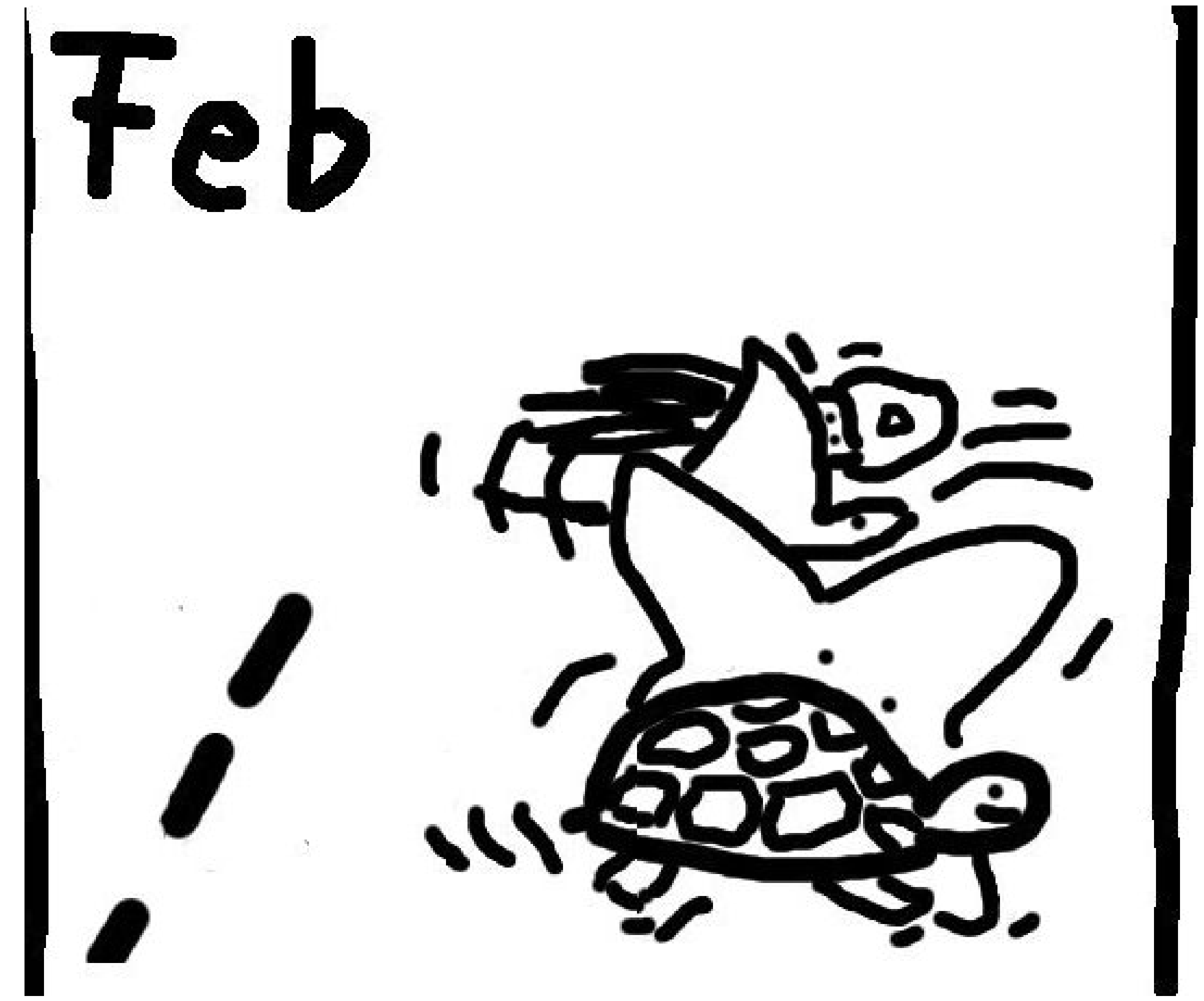
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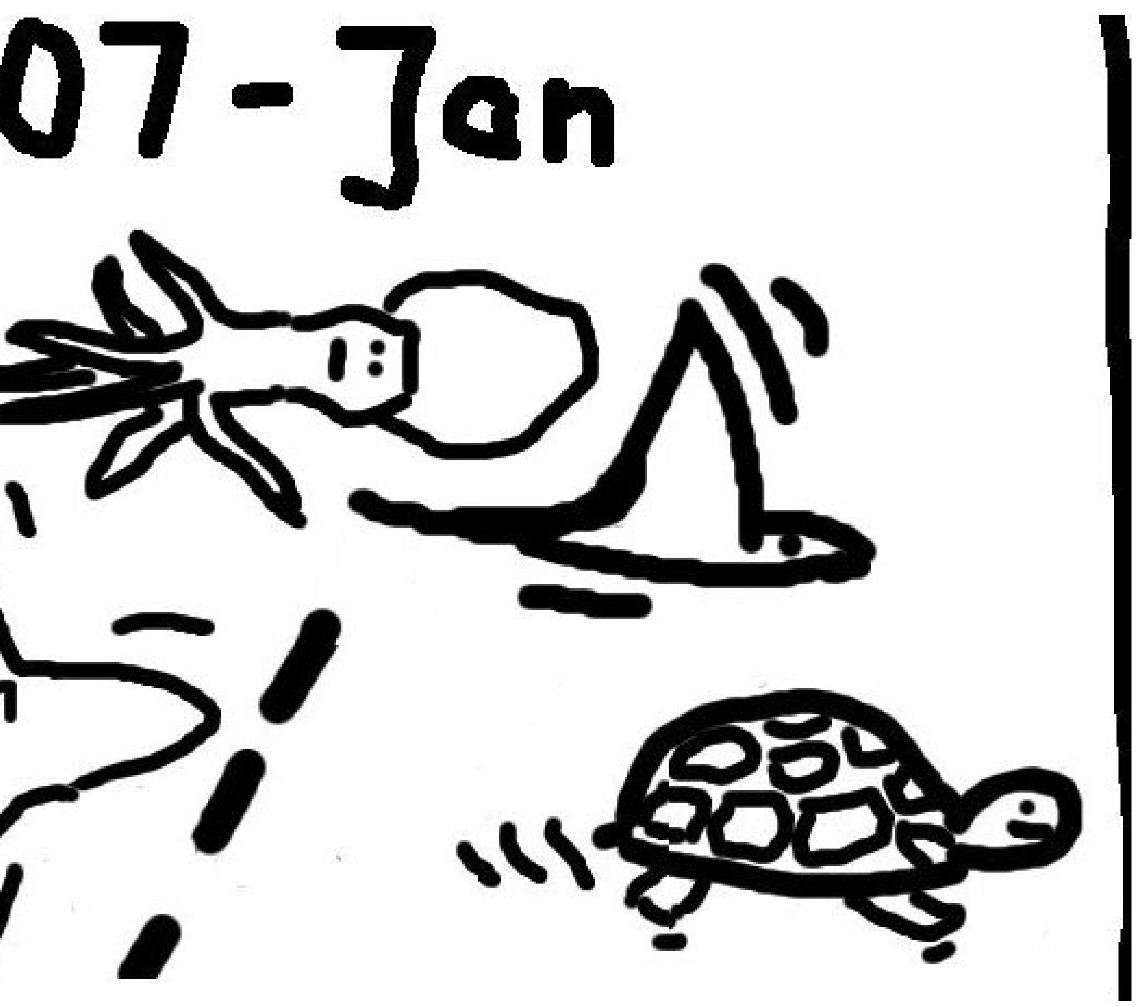
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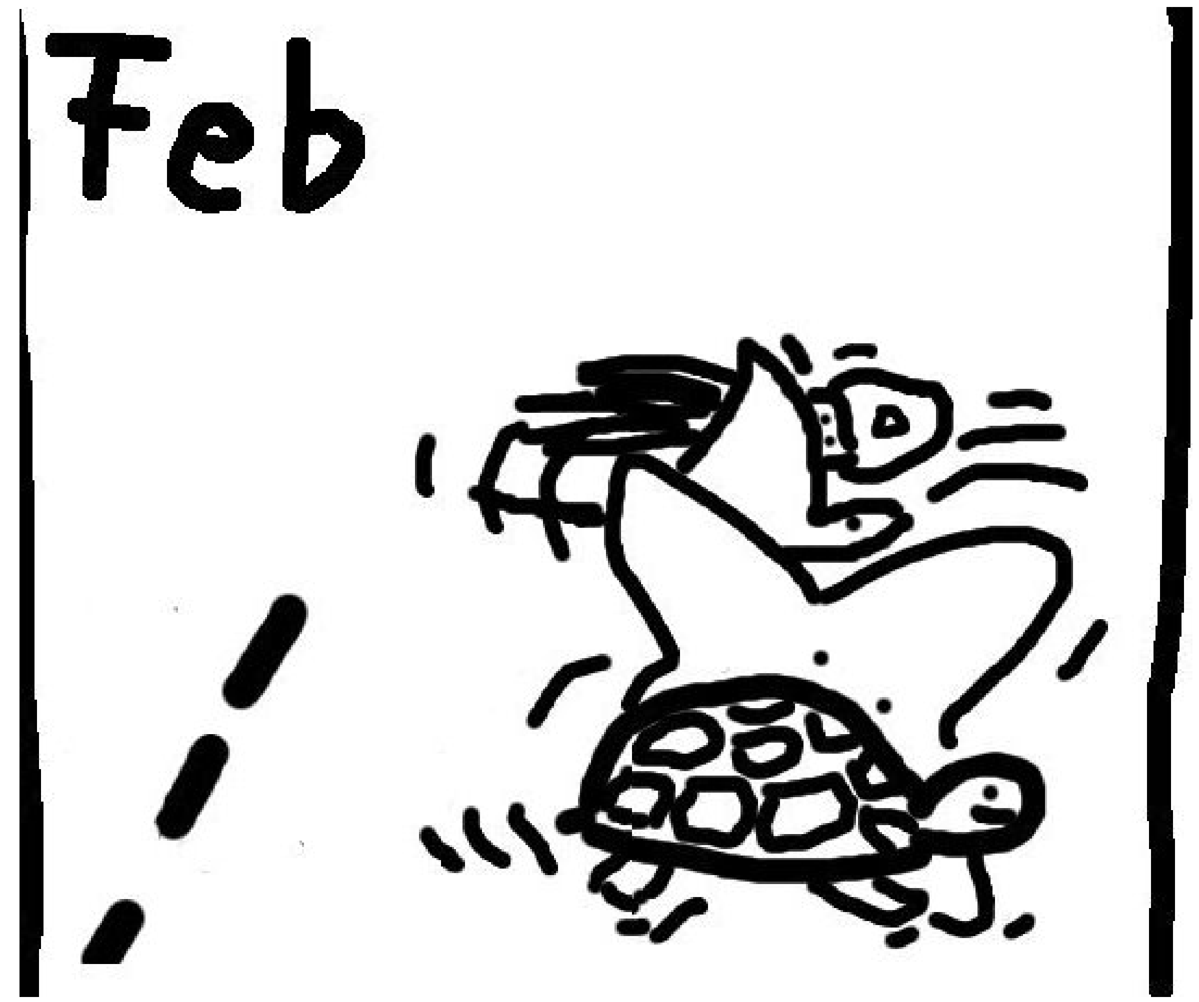
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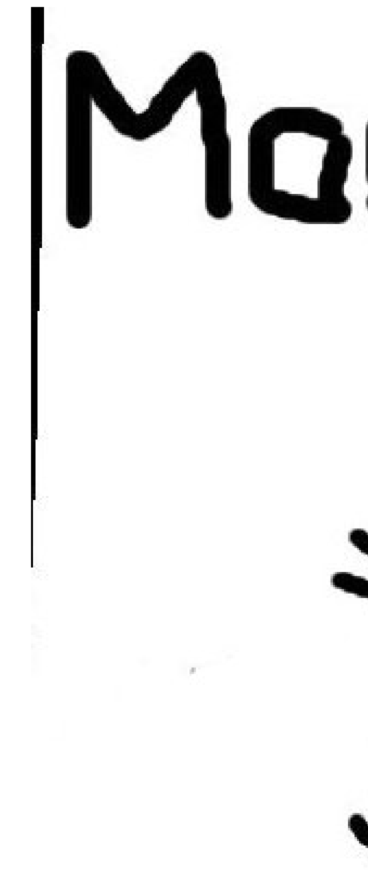

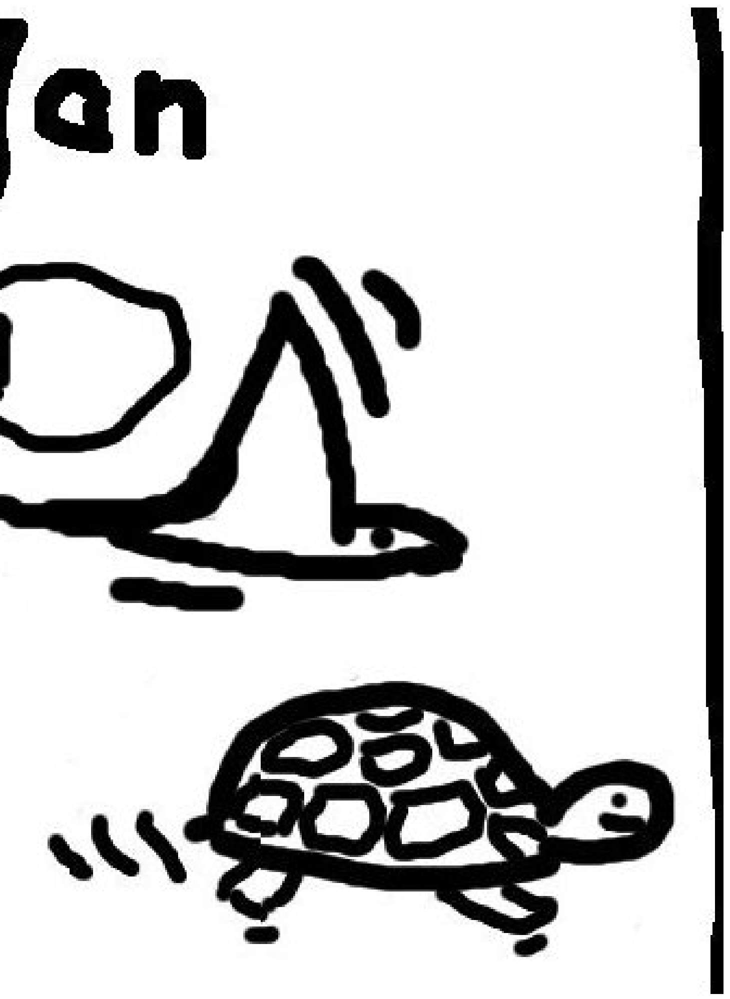
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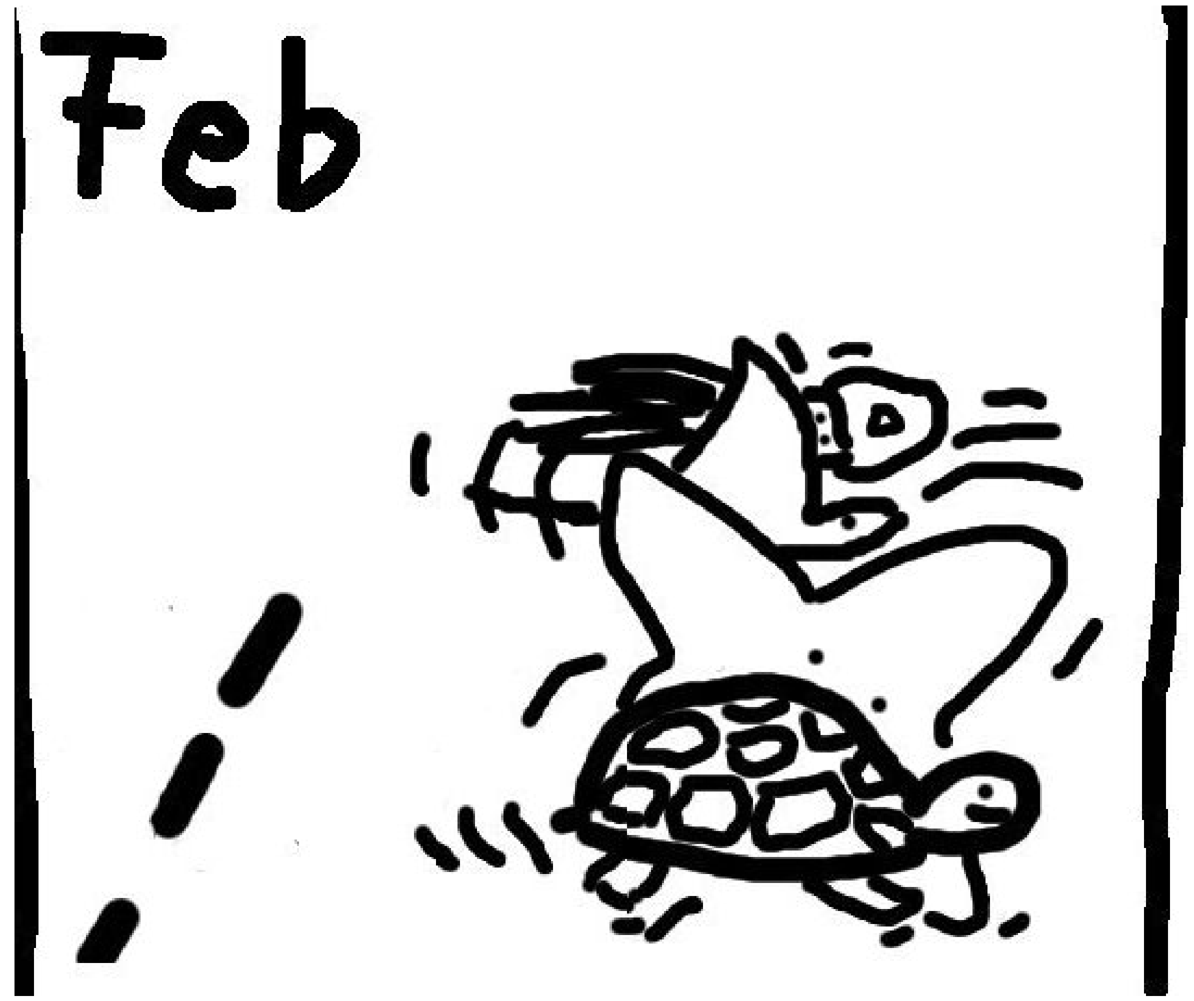
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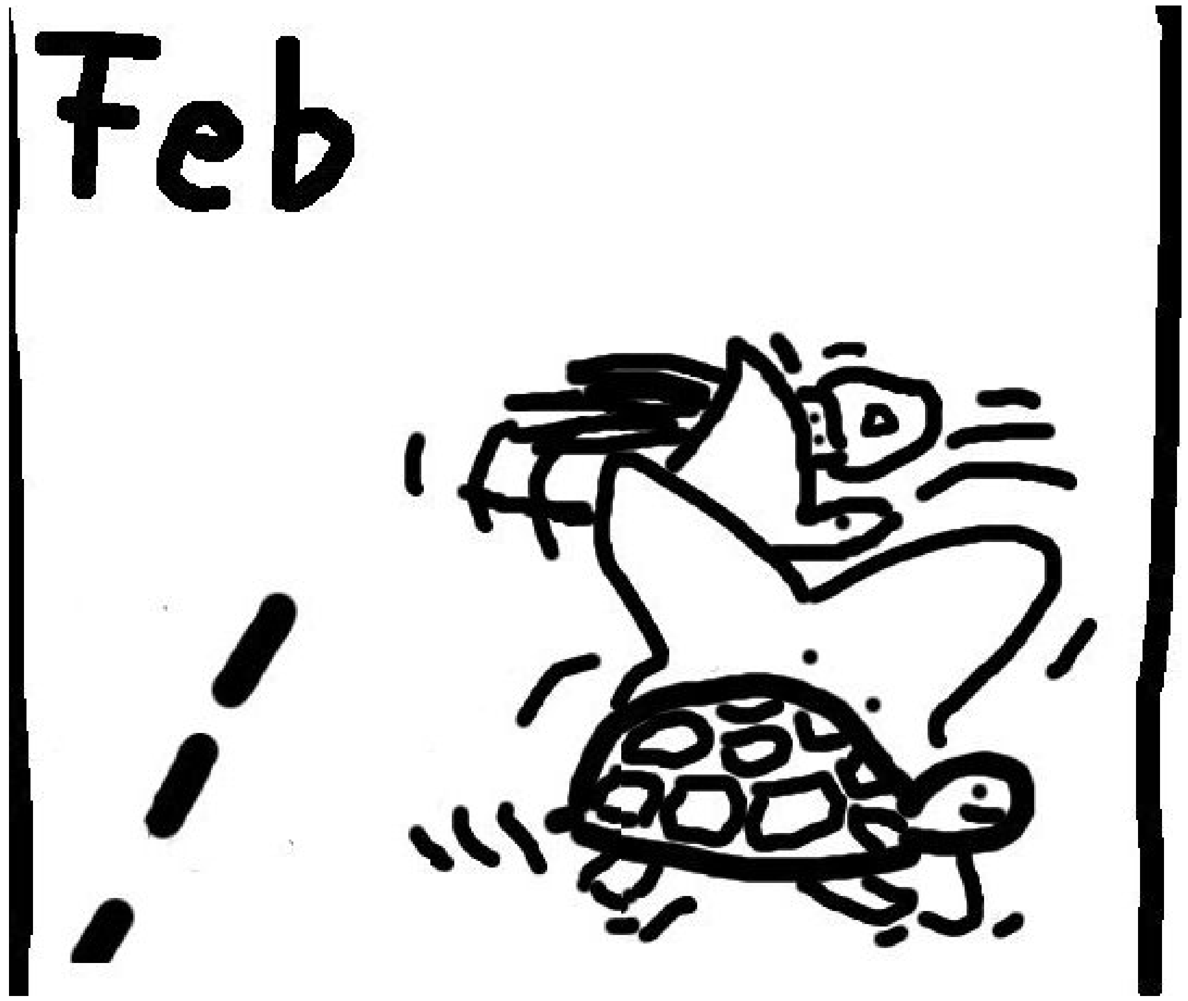
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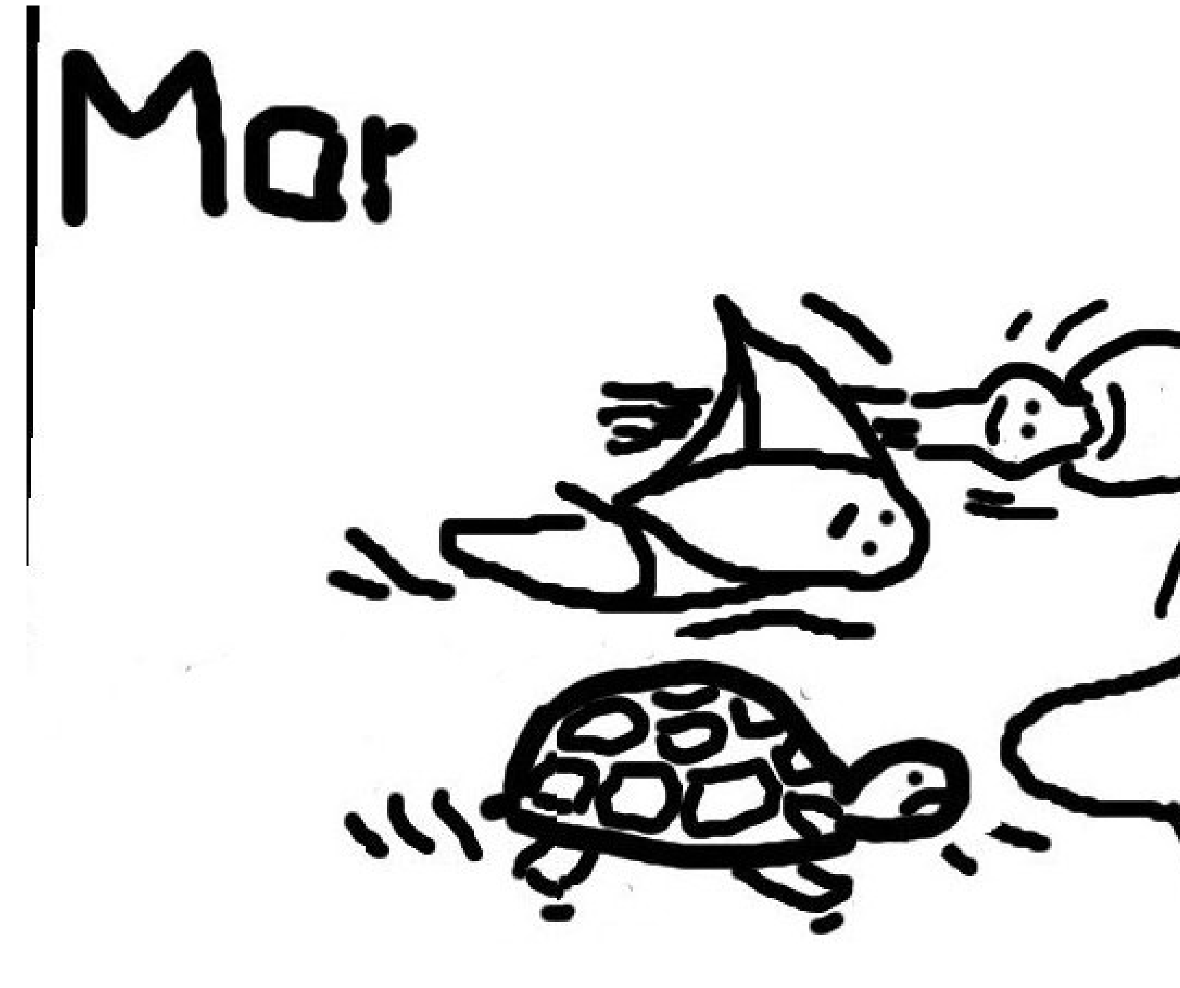
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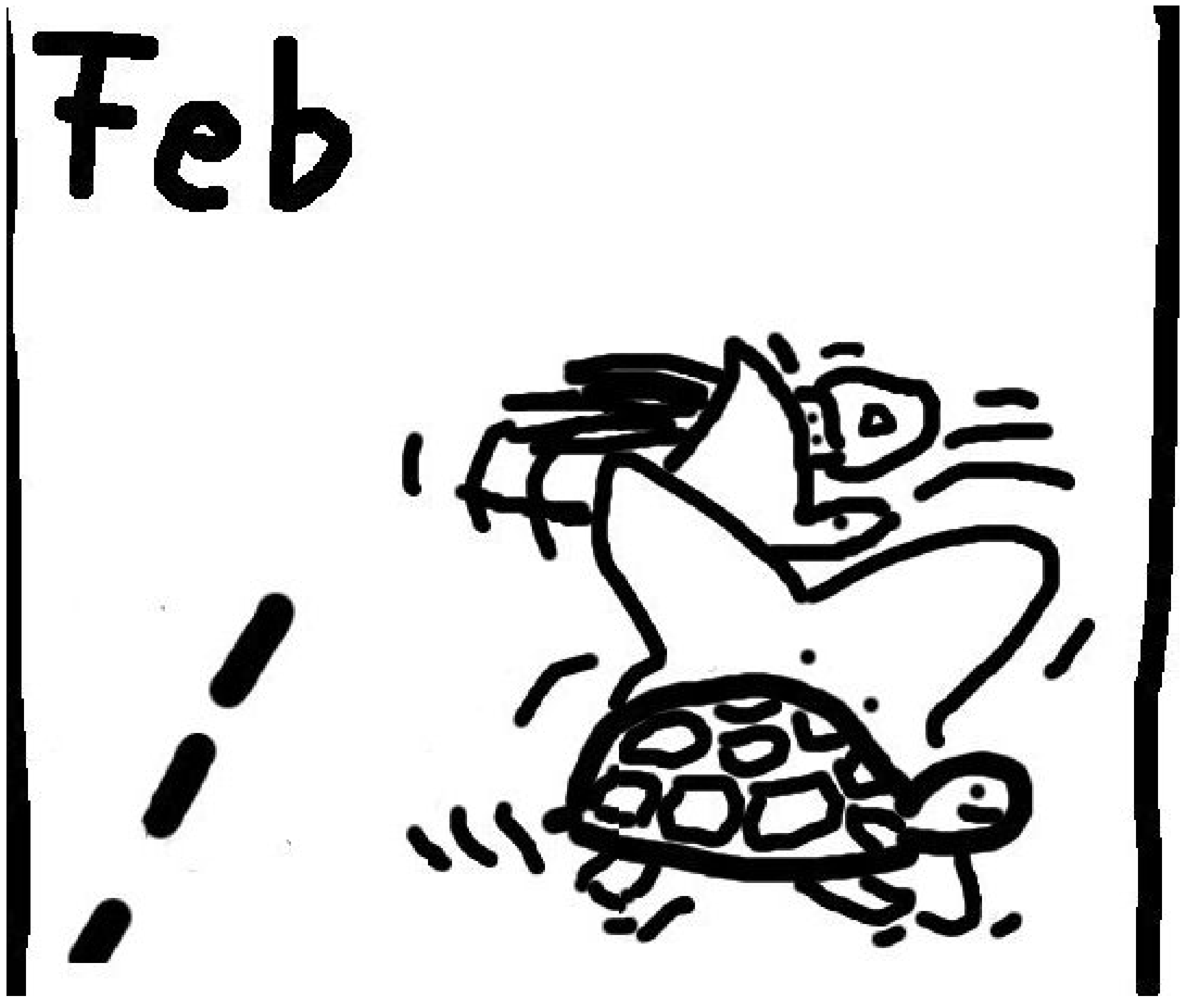
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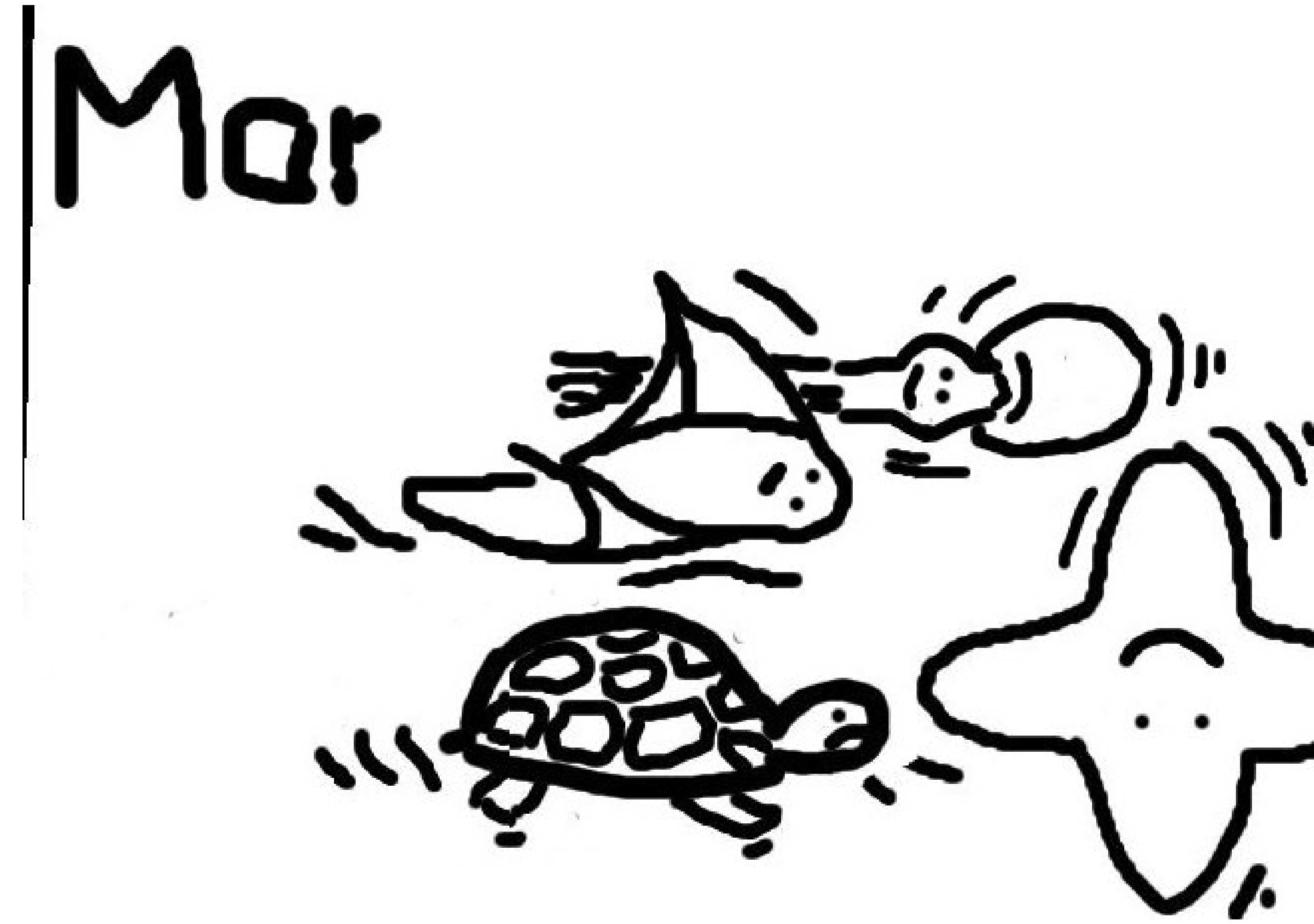
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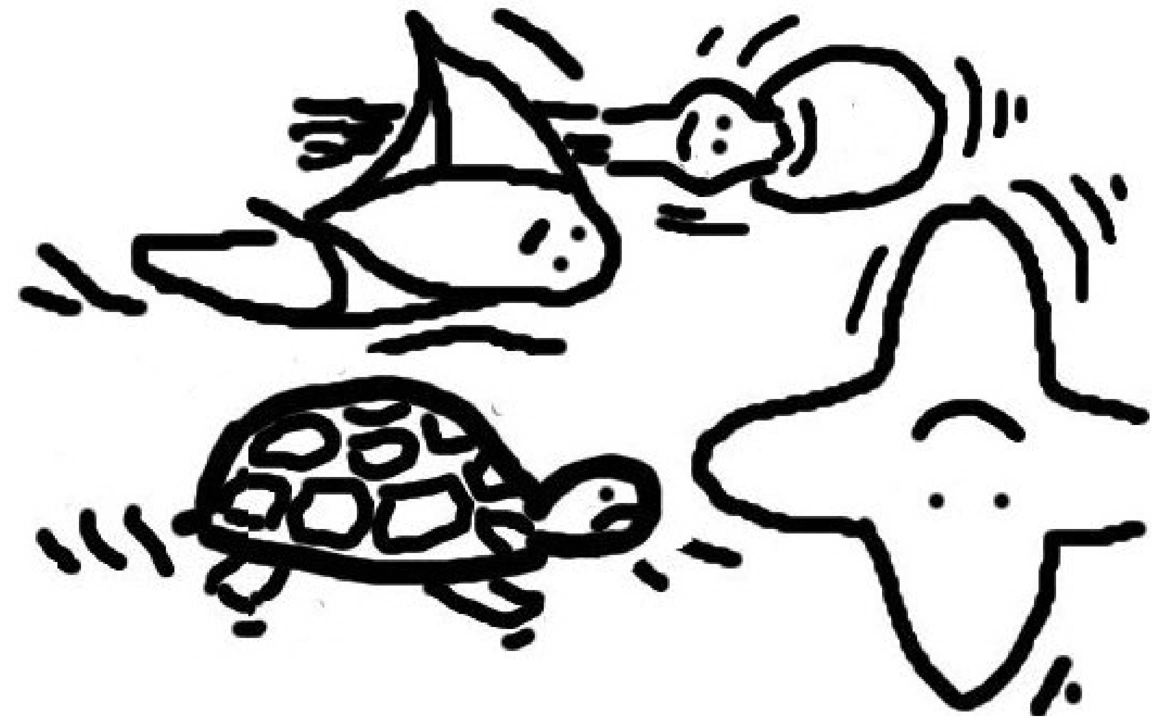

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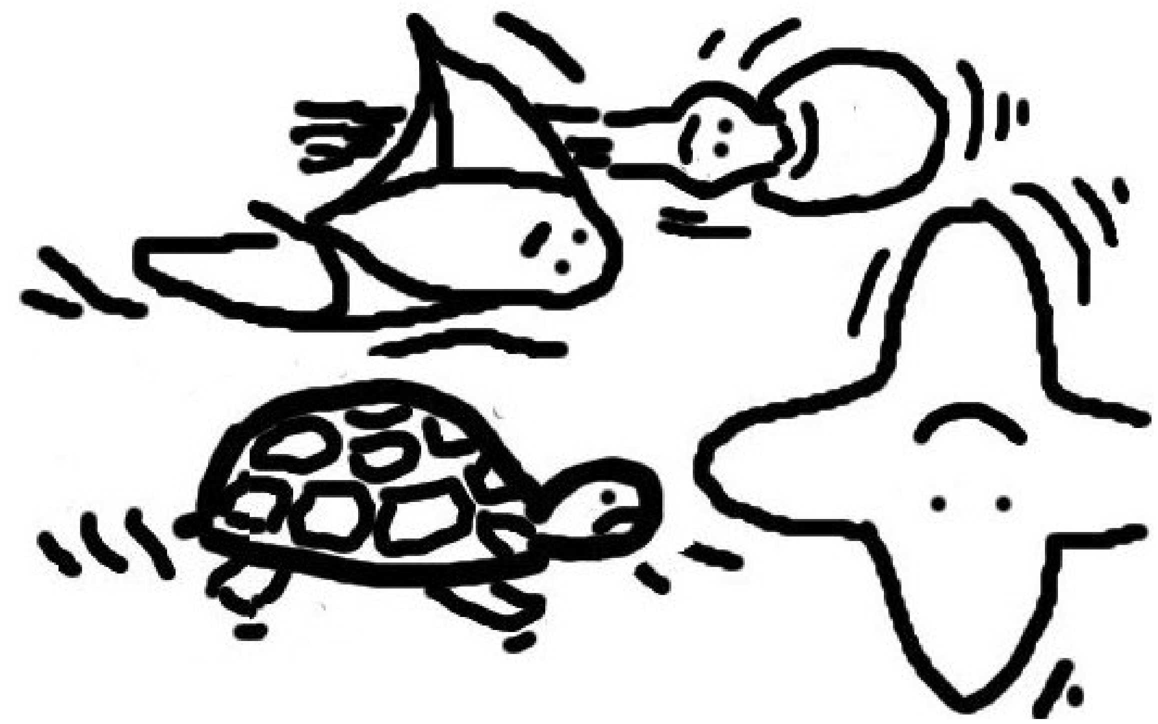

Zoom

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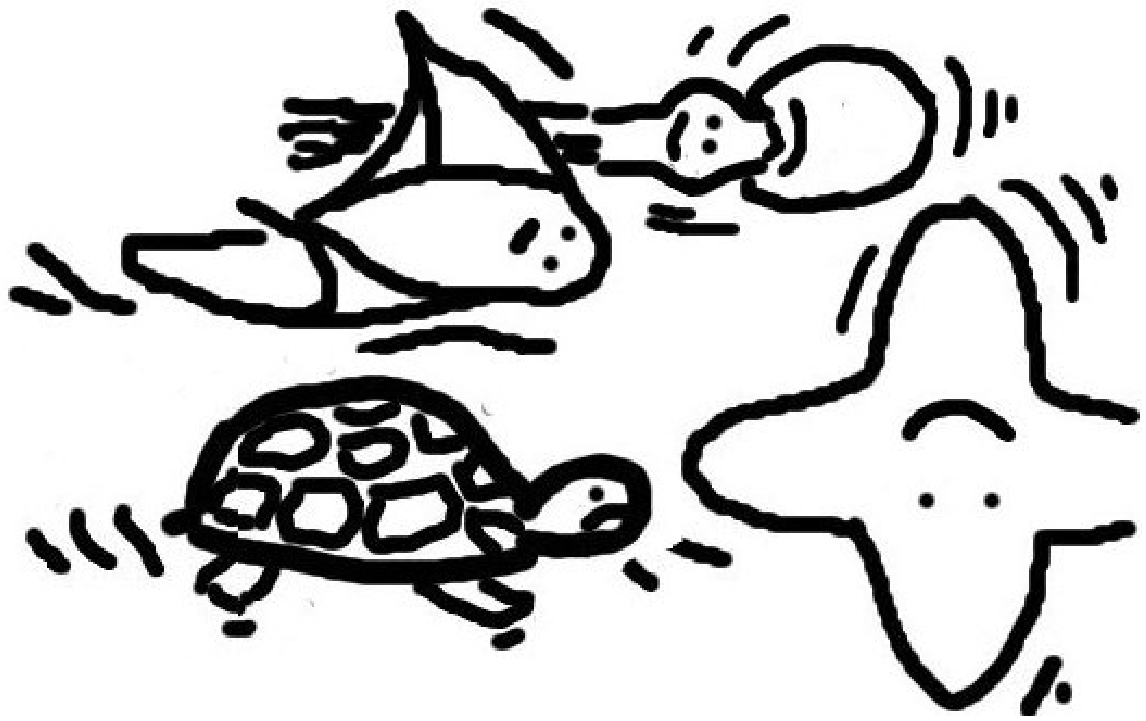
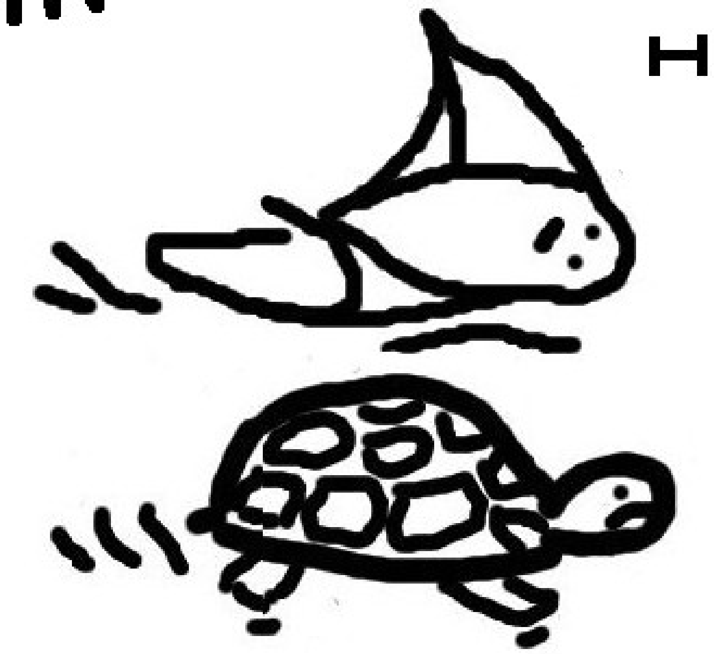


Zoom



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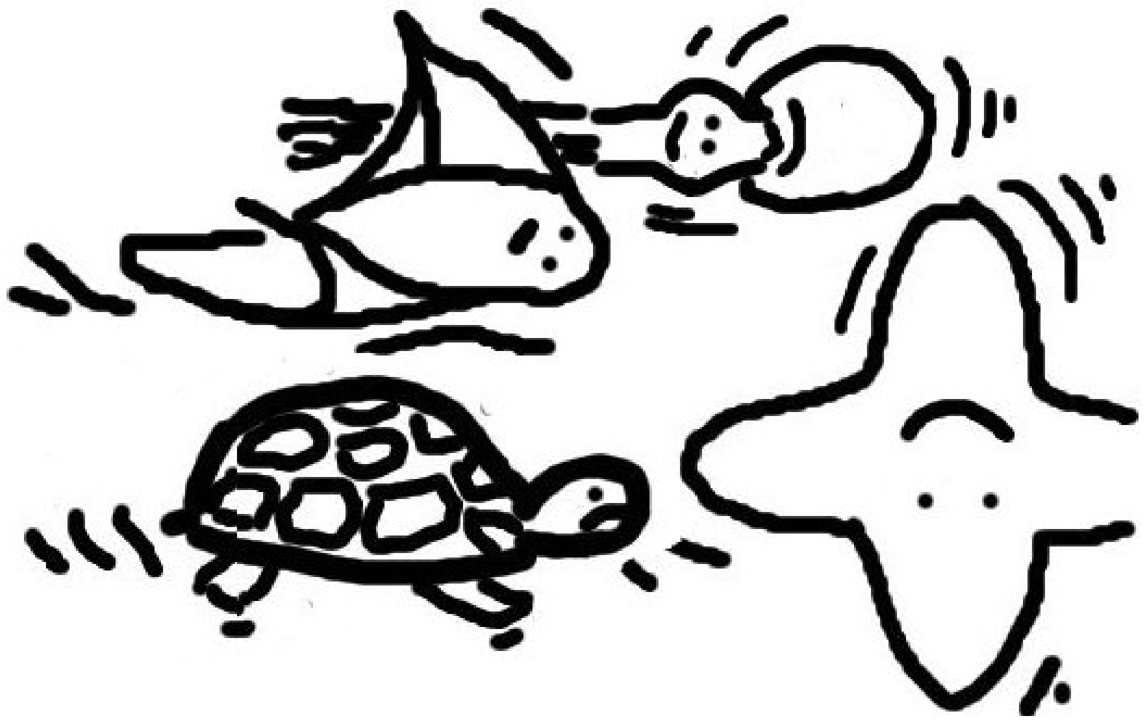
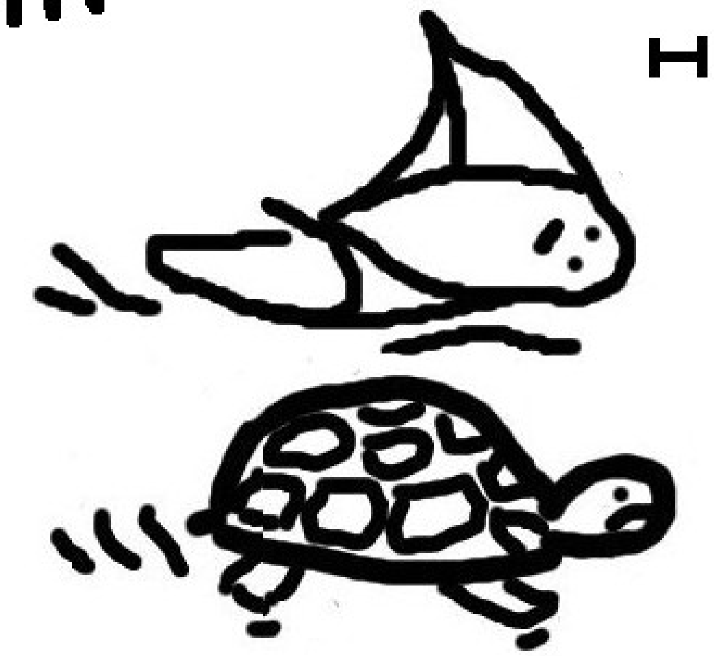


Zoom



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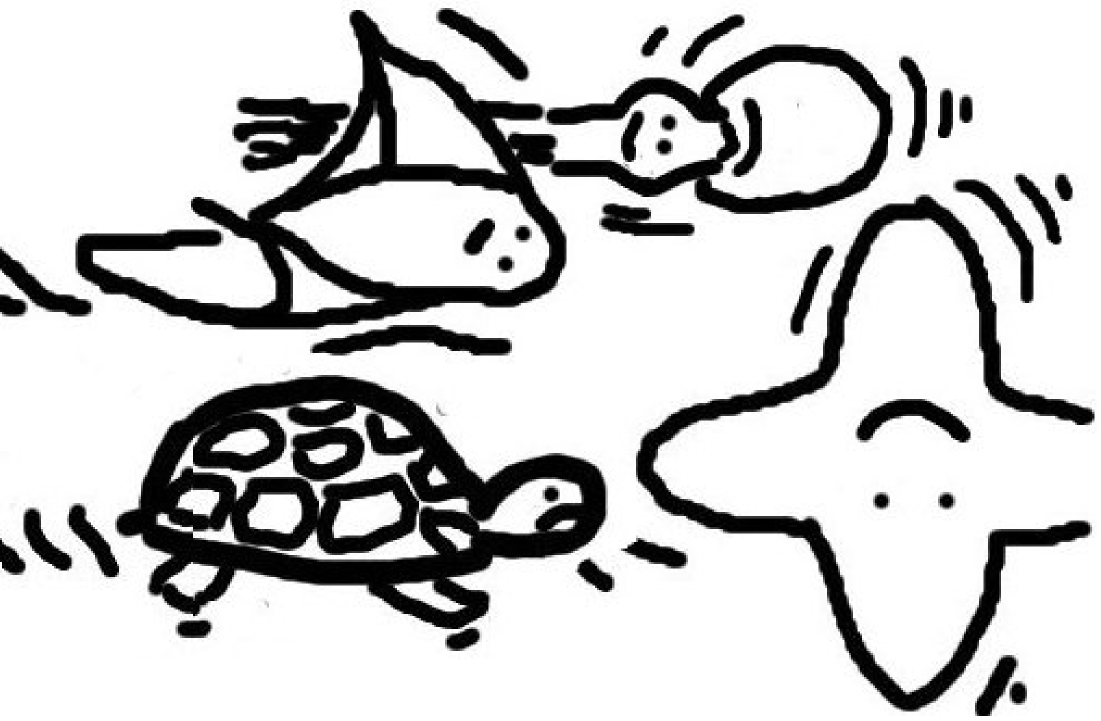
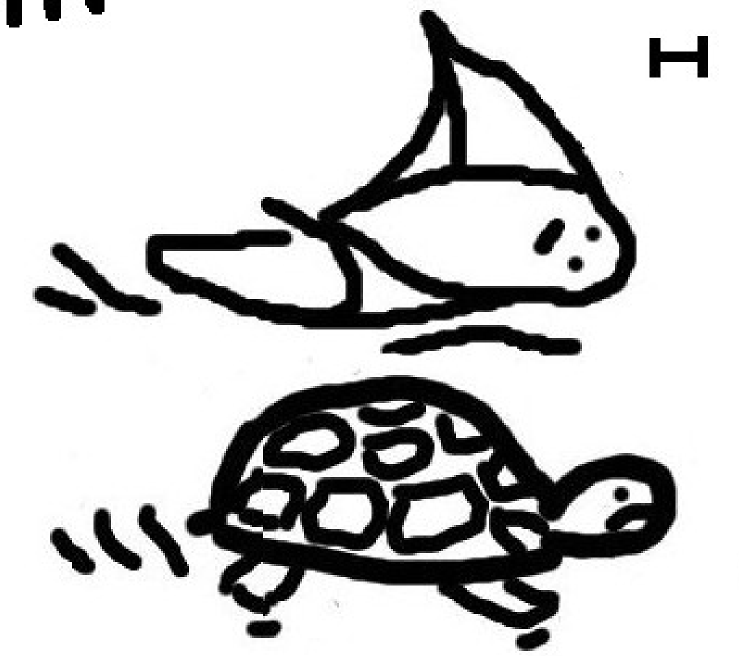


Zoom



Faster H

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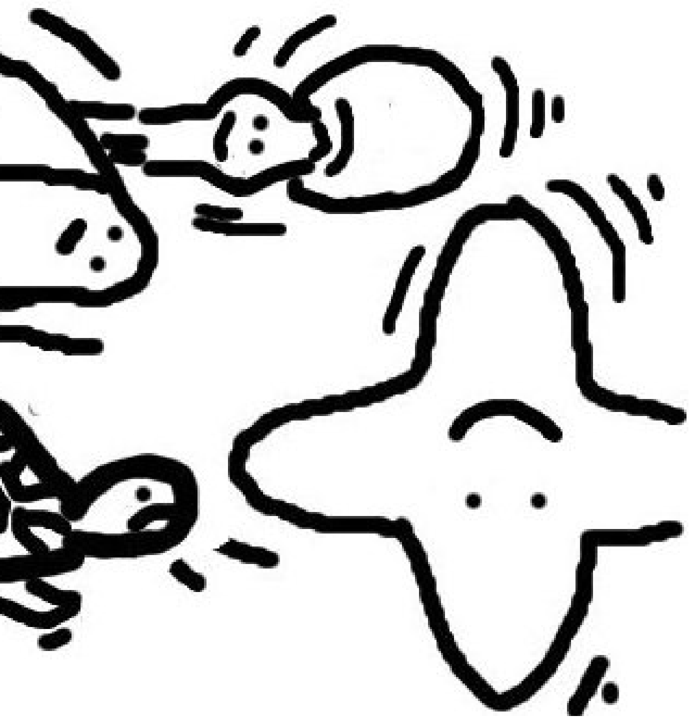
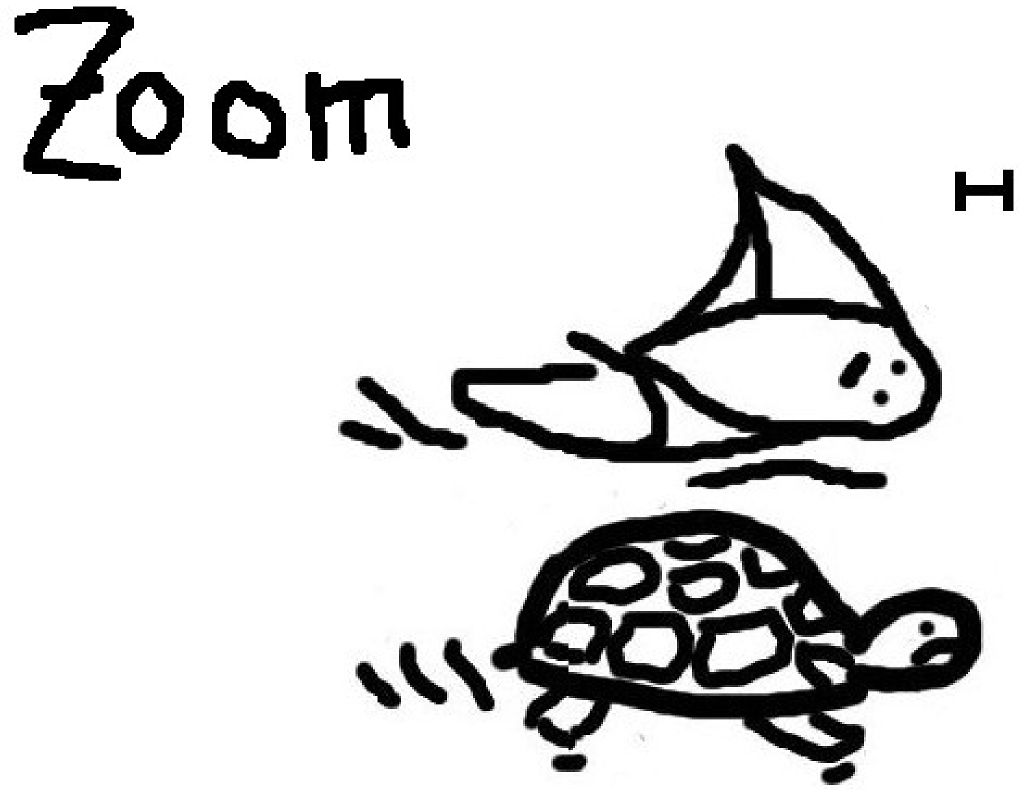
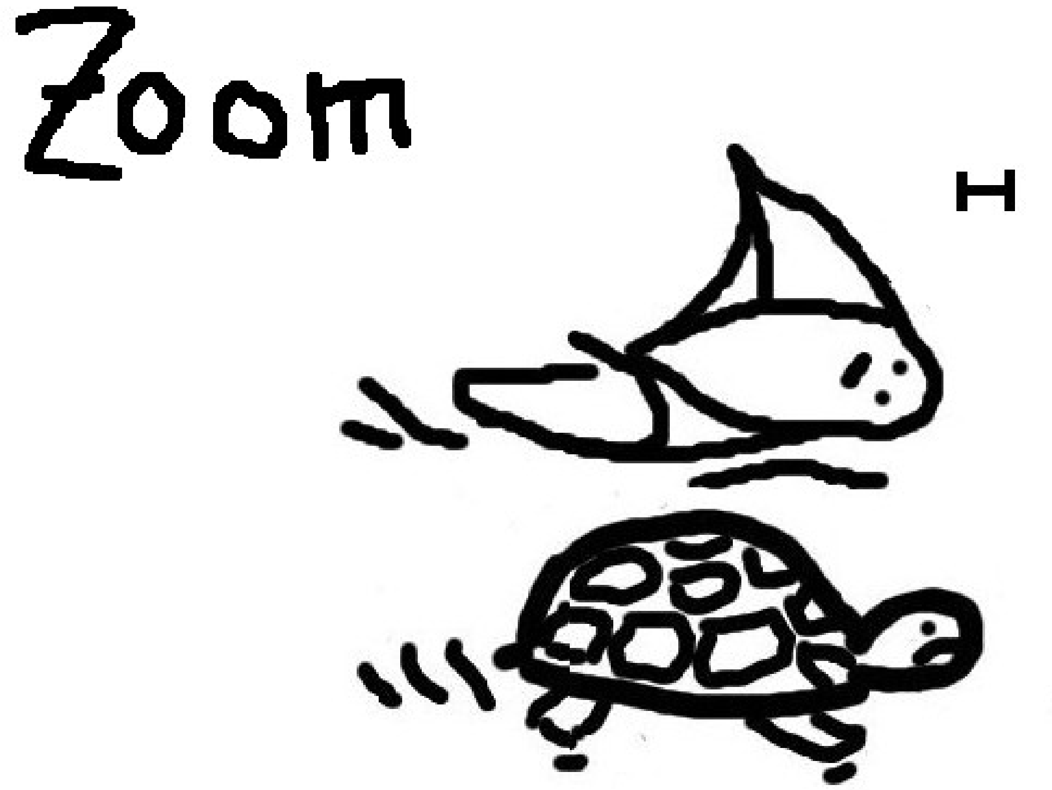




Faster Hessian arit

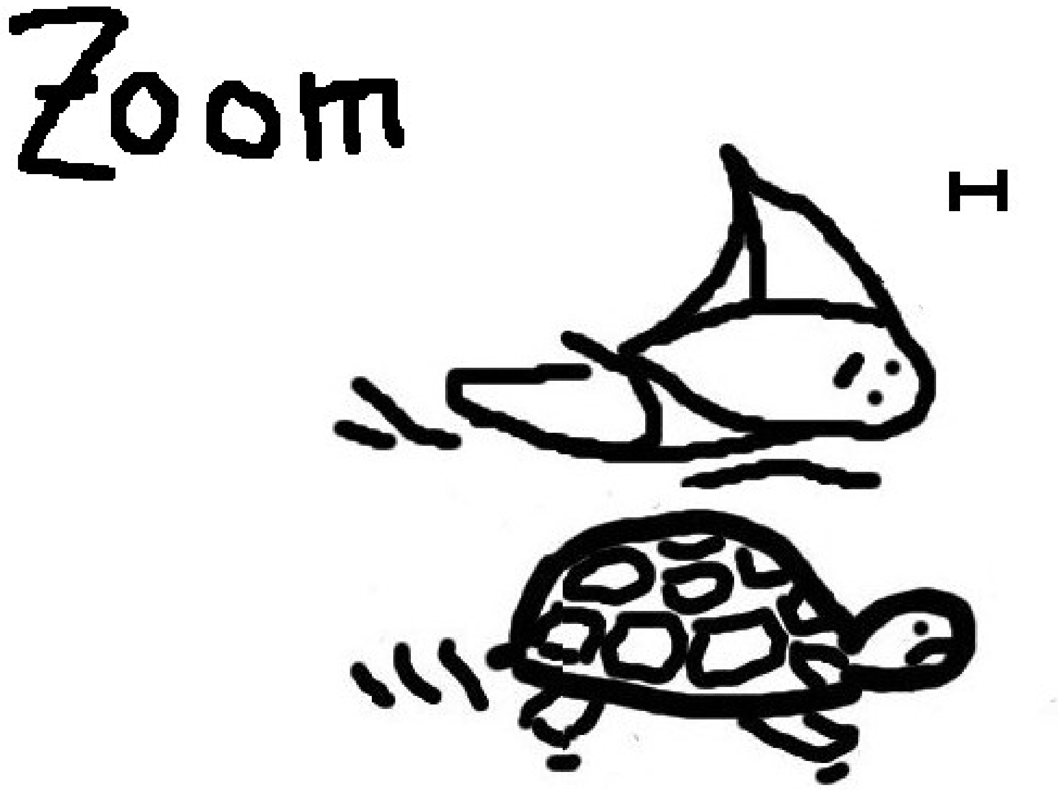
2007 Hisil–Carter–

7.8**M** for DBL.



Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.



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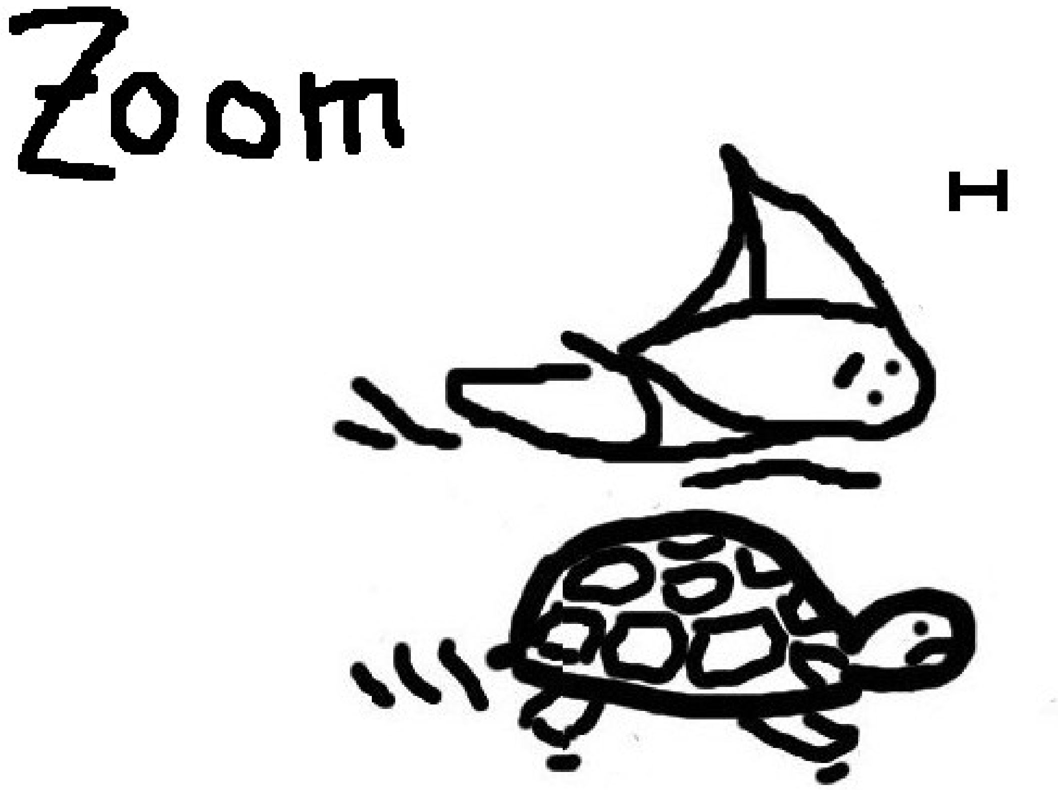

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
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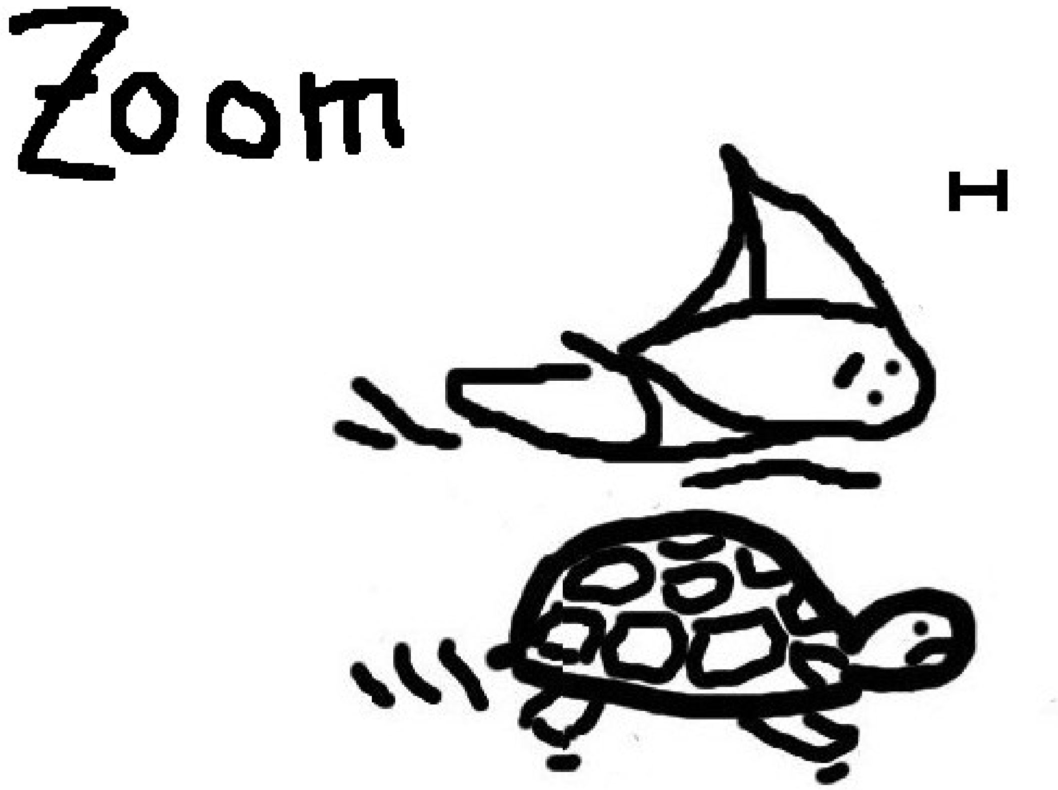
2010 Hisil: 11**M** for ADD.



Faster Hessian arithmetic

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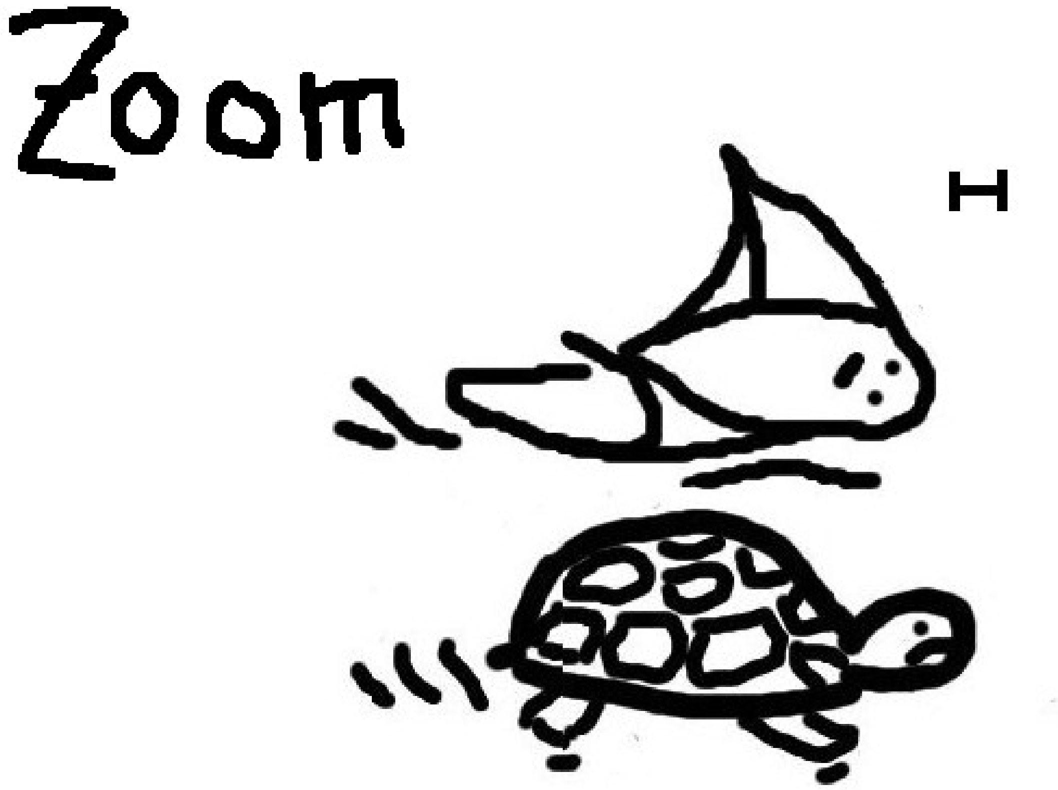
2010 Hisil: 11**M** for ADD.

Hessian tied with Weierstrass for DBL-DBL-DBL-DBL-DBL-ADD.

Need to zoom in closer: analyze exact **S**/**M**, overhead for checking for special cases, extra DBL, extra ADD, etc.



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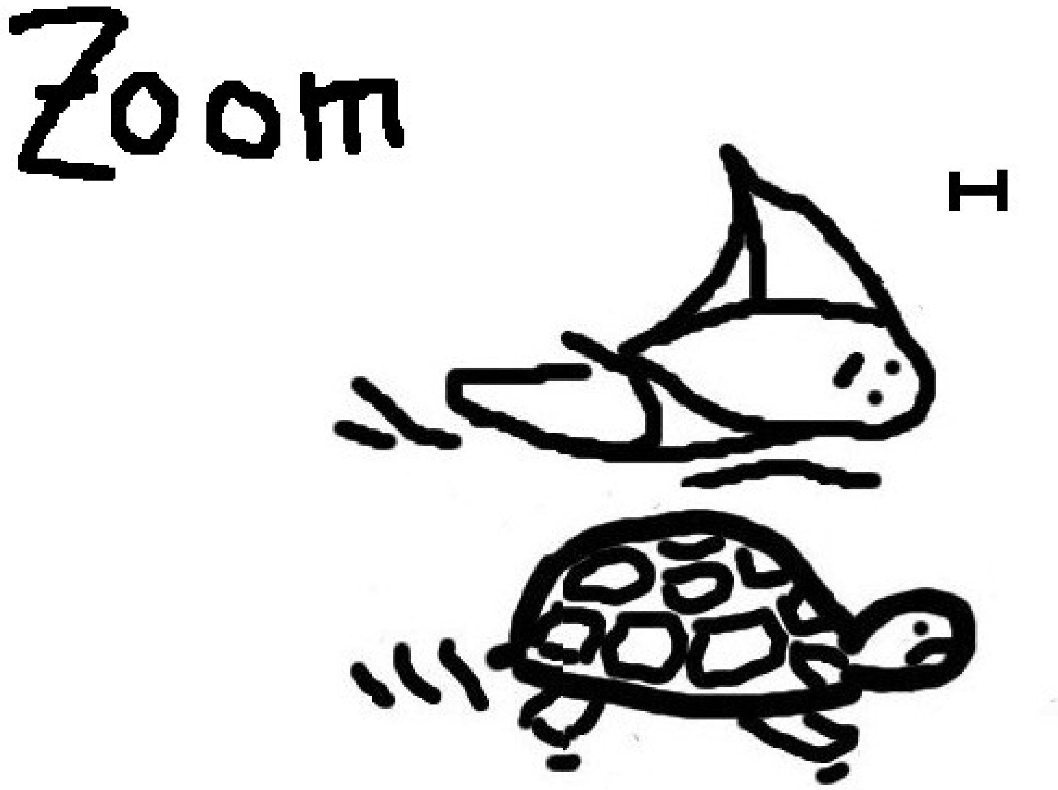
Or speed up Hessian more.



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Faster Hessian arithmetic

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with $a(2$

2007 7.8
2010 11
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Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
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2010 Hisil: 11**M** for ADD.

Hessian tied with Weierstrass for
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New: 7.6**M** for DBL.

New (announced J

Generalize to more
twisted Hessian c
$aX^3 + Y^3 + Z^3 =$
with $a(27a - d^3)$

2007 7.8**M** DBL id
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Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

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New: 7.6**M** for DBL.

New (announced July 2009)

Generalize to more curves: **twisted Hessian curves** $aX^3 + Y^3 + Z^3 = dXYZ$ with $a(27a - d^3) \neq 0$.

2007 7.8**M** DBL idea fails, b
2010 11**M** ADD generalizes,
new 7.6**M** DBL generalizes.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

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2007 7.8**M** DBL idea fails, but 2010 11**M** ADD generalizes, new 7.6**M** DBL generalizes.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

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2007 7.8**M** DBL idea fails, but
2010 11**M** ADD generalizes,
new 7.6**M** DBL generalizes.

Rotate addition law
so that it also works for DBL;
complete if a is not a cube.
Eliminates special-case overhead,
helps stop side-channel attacks.

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zoom in closer:
exact **S**/**M**, overhead
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Triplings

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Triplings (assumin

TPL is $P \mapsto 3P$.

2007 Hisil–Carter–
12.8**M** for Hessian

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New (announced July 2009):

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twisted Hessian curves
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Triplings (assuming $d \neq 0$)

TPL is $P \mapsto 3P$.

2007 Hisil–Carter–Dawson:
12.8**M** for Hessian TPL.

Generalizes to twisted Hessia

New (announced July 2009):

Generalize to more curves:

twisted Hessian curves

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12.8**M** for Hessian TPL.

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2015 Kohel: 11.2**M**.

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TPL is $P \mapsto 3P$.

2007 Hisil–Carter–Dawson:
12.8**M** for Hessian TPL.

Generalizes to twisted Hessian.

2015 Kohel: 11.2**M**.

New: 10.8**M** assuming
field with fast primitive $\sqrt[3]{1}$;
e.g., $\mathbf{F}_q[\omega]/(\omega^2 + \omega + 1)$, or
$\mathbf{F}_p$ with $7p = 2^{298} + 2^{149} + 1$.

(More history in small char.
See paper for details.)

nnounced July 2009):

ze to more curves:

Hessian curves

$Y^3 + Z^3 = dXYZ$

$27a - d^3) \neq 0$.

3**M** DBL idea fails, but

M ADD generalizes,

M DBL generalizes.

addition law

't also works for DBL;

te if a is not a cube.

es special-case overhead,

op side-channel attacks.

<u>Triplings (assuming $d \neq 0$)</u>

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If $aX^3 +$

then VW

where

$U = -X$

If $VW(V$

then aX

where Q

$S = -(V$

$dX_3 = F$

$Y_3 = RS$

$Z_3 = RV$

Compose

$(X_3 : Y_3$

July 2009):

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curves

$= dXYZ$

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If $aX^3 + Y^3 + Z^3$
then $VW(V + dU$
where
$U = -XYZ$, $V =$

If $VW(V + dU +$
then $aX_3^3 + Y_3^3 +$
where $Q = dU$, R
$S = -(V + Q + R$
$dX_3 = R^3 + S^3 +$
$Y_3 = RS^2 + SV^2$
$Z_3 = RV^2 + SR^2$

Compose these 3-i
$(X_3 : Y_3 : Z_3) = 3$

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If $aX^3+Y^3+Z^3=dXYZ$
then $VW(V+dU+aW)=$
where
$U=-XYZ$, $V=Y^3$, $W=$

If $VW(V+dU+aW)=U^3$
then $aX_3^3+Y_3^3+Z_3^3=dX_3$
where $Q=dU$, $R=aW$,
$S=-(V+Q+R)$,
$dX_3=R^3+S^3+V^3-3RS$
$Y_3=RS^2+SV^2+VR^2-3$
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Compose these 3-isogenies:
$(X_3:Y_3:Z_3)=3(X:Y:Z$

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For three
(α,β,γ)
$(\alpha R+\beta$
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$+ (\beta\alpha^2$
$+ (\alpha+\beta$

Also use
Solve for

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Three cubings for

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Also use $a(R + S +$
Solve for $dX_3, Y_3,$

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Three cubings for R, S, V.

For three choices of constan
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$(\alpha R + \beta S + \gamma V) \cdot$
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$-Y^3 + Z^3 = dXYZ$

$V(V + dU + aW) = U^3$

XYZ, $V = Y^3$, $W = X^3$.

$/ + dU + aW) = U^3$

$^3_3 + Y_3^3 + Z_3^3 = dX_3Y_3Z_3$

$= dU$, $R = aW$,

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$R^3 + S^3 + V^3 - 3RSV$,

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2015 Ko

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(α, β, γ)

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$= dXYZ$

$+ aW) = U^3$

Y^3, $W = X^3$.

$aW) = U^3$

$Z_3^3 = dX_3Y_3Z_3$

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2015 Kohel's 11.2

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$(\alpha, \beta, \gamma) = (1, 1, 1$

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2015 Kohel's 11.2**M**
(4 cubings + 4 mults)
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New 10.8**M** (6 cubings) makes faster choices assuming fast primitive $\omega = \sqrt[3]{1}$:
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Are triplings usefu

2005 Dimitrov–Im
"double-base chai
compute 314159P
$2^{15}3^2P + 2^{11}3^2P$
$+ 2^4 3^1 P - 2$
2TPL, 15DBL, 4A

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e.g., compute 314
$2^{12}3^3 3P - 2^7 3^3 5P$
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Are triplings useful?

2005 Dimitrov–Imbert–Mish
"double-base chains": e.g.,
compute $314159P$ as
$$2^{15}3^2P + 2^{11}3^2P + 2^8 3^1 P$$
$$+ 2^4 3^1 P - 2^0 3^0 P.$$
2TPL, 15DBL, 4ADD.

2006 Doche–Imbert
generalized double-base chai
e.g., compute $314159P$ as
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after precomputing $3P, 5P, 7$
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$$2^{15}3^2P + 2^{11}3^2P + 2^83^1P + 2^43^1P - 2^03^0P.$$

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$$2^{12}3^33P - 2^73^35P - 2^43^17P - 2^03^0P$$

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3TPL, 13DBL, 6ADD.

Not good for cons

Good for signature

factorization, matl

Also need time to

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Not good for constant time.
Good for signature verificati
factorization, math, etc.

Also need time to compute
Good for scalars used many

Are triplings useful?

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$$2^{12}3^3 3P - 2^7 3^3 5P - 2^4 3^1 7P - 2^0 3^0 P$$

after precomputing $3P, 5P, 7P$.
3TPL, 13DBL, 6ADD.

Not good for constant time.
Good for signature verification, factorization, math, etc.

Also need time to compute chain.
Good for scalars used many times.

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2005 Dimitrov–Imbert–Mishra "double-base chains": e.g., compute $314159P$ as

$$2^{15}3^2P + 2^{11}3^2P + 2^83^1P + 2^43^1P - 2^03^0P.$$

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Double-base chains speed up Weierstrass curves slightly: 9.29**M**/bit for 256-bit scalars.

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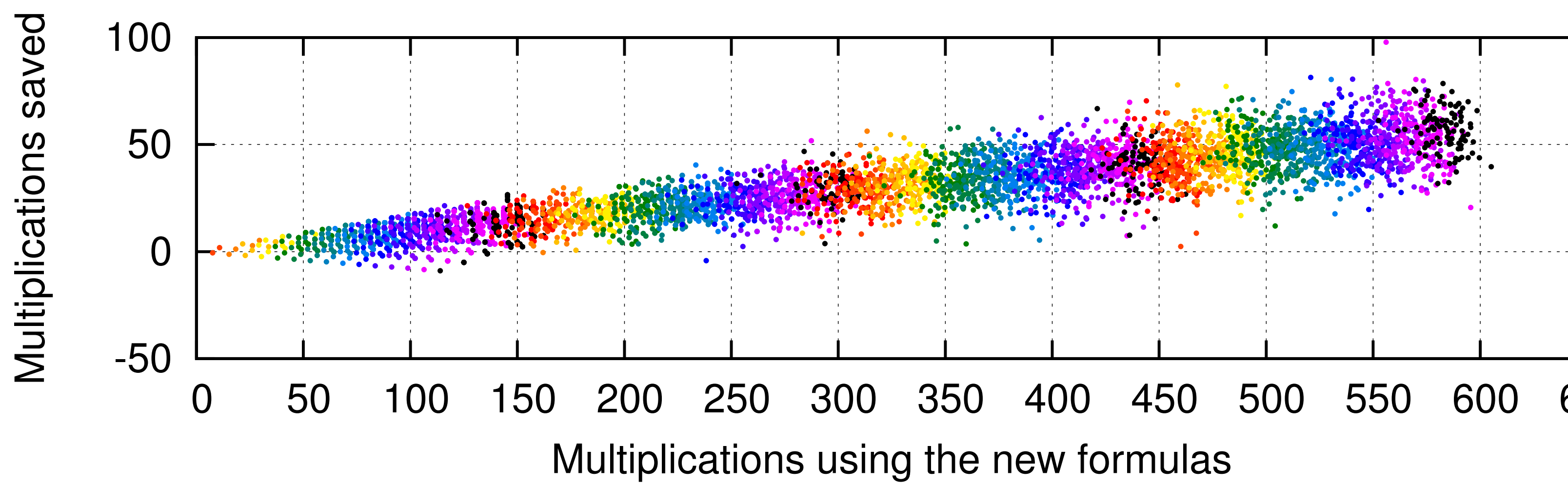
Comparison to Weierstrass for 1-bit, 2-bit, . . . , 64-bit scalars:



Uses 2008 Doche–Habsieger "tree search" and some new improvements: e.g., account for costs of ADD, DBL, TPL.

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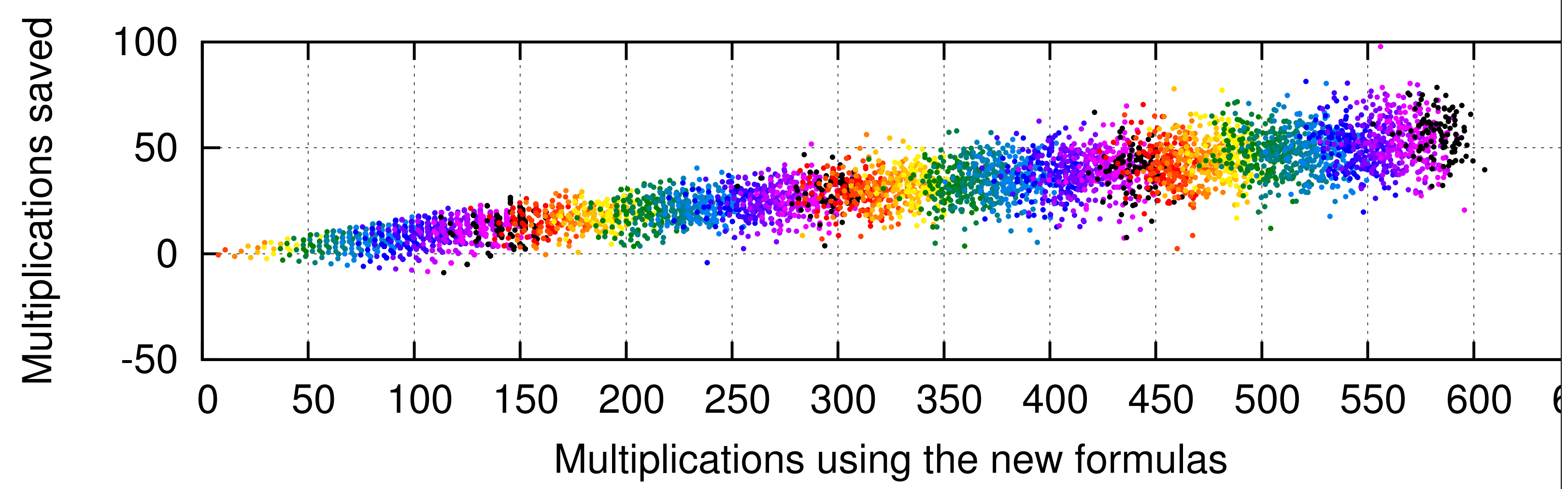

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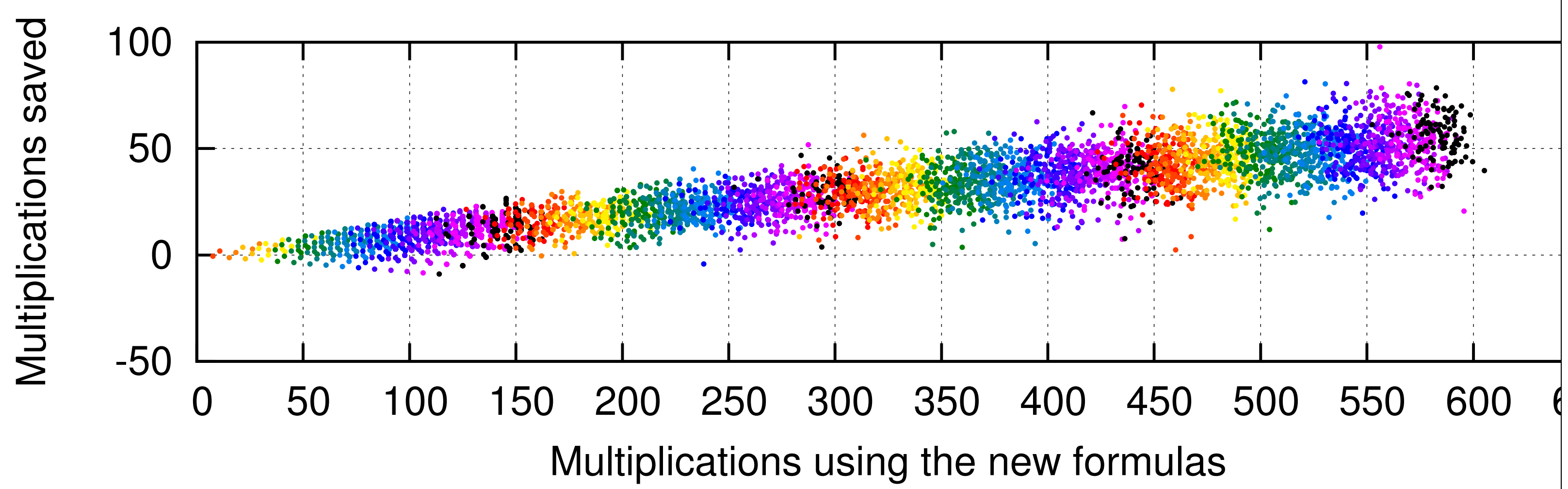

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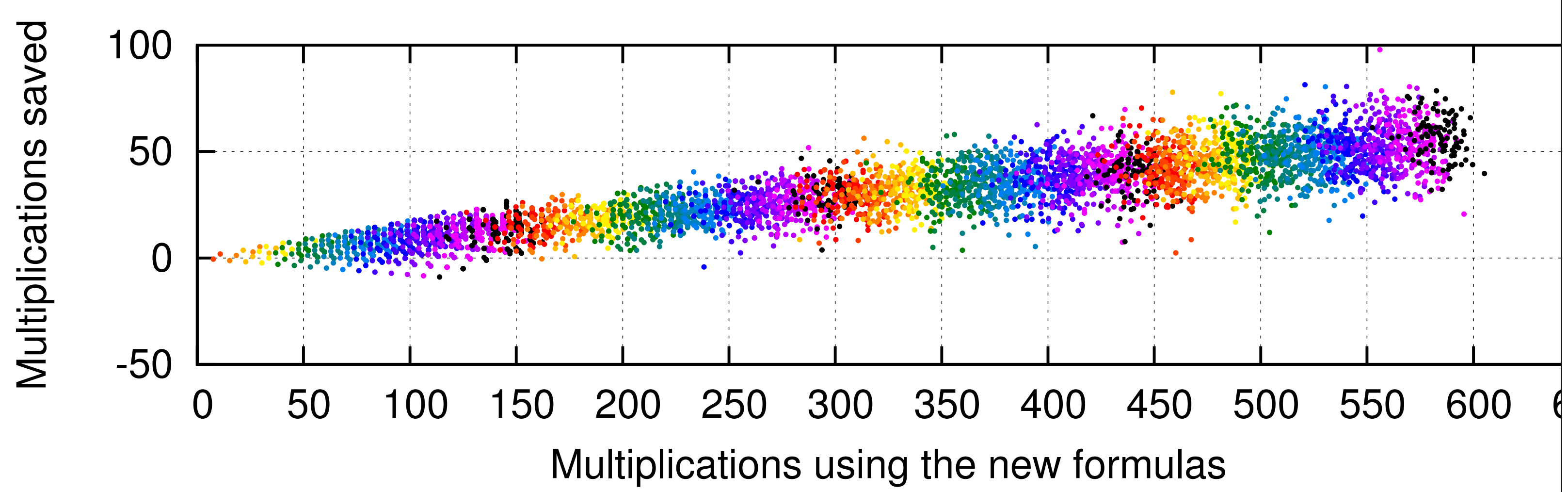

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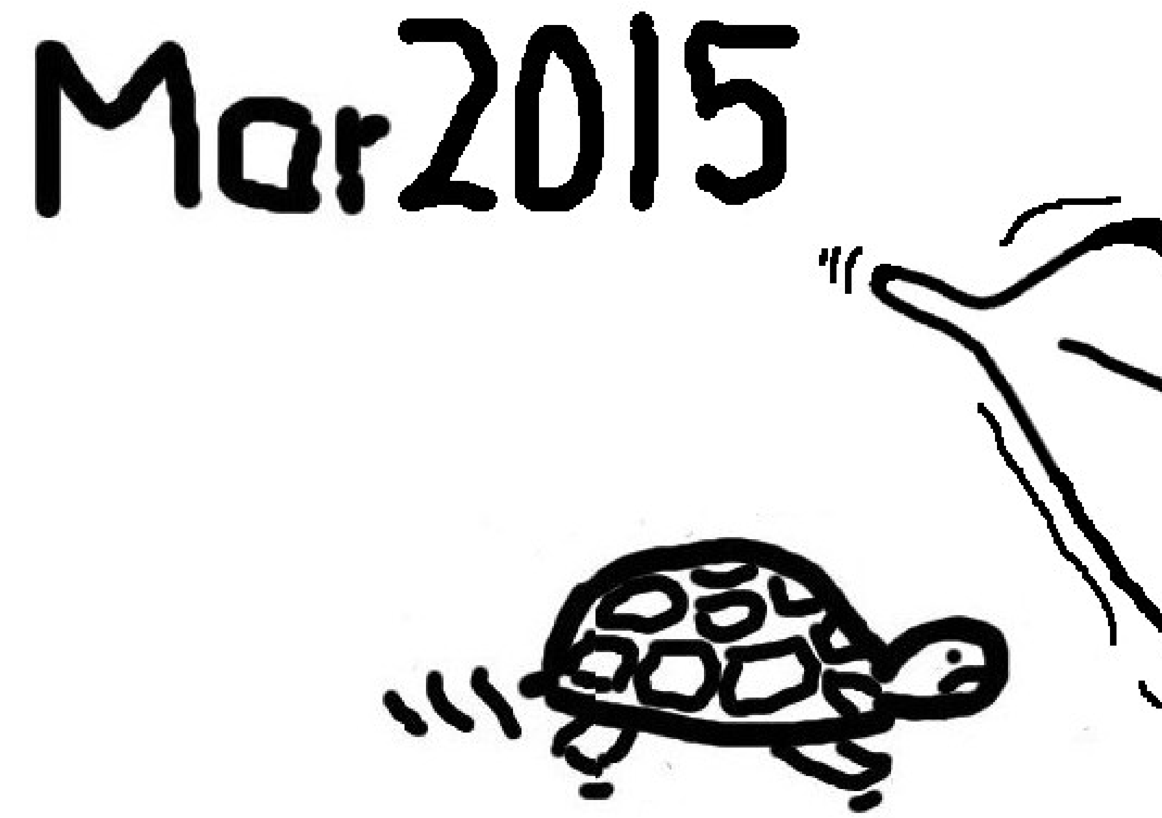

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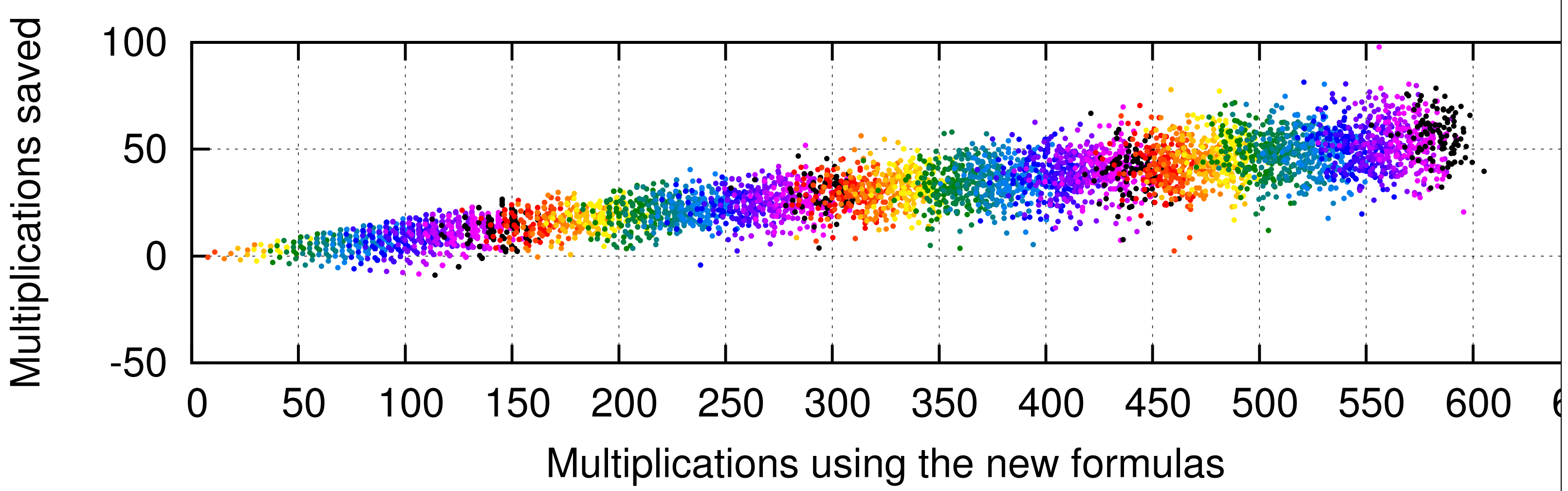

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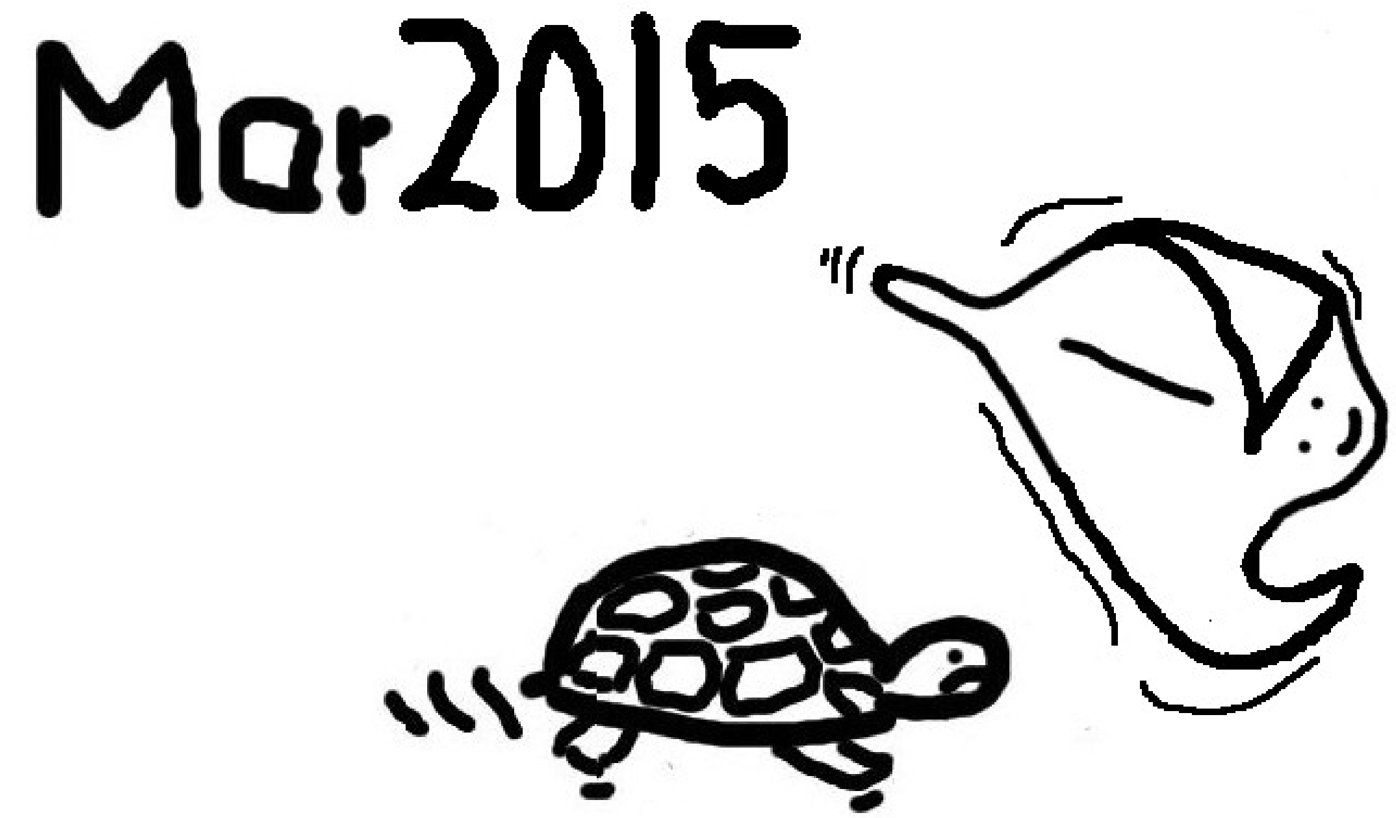

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