Exploring the parameter space in lattice attacks

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Based on attack survey from 2019 Bernstein–Chuengsatiansup–Lange–van Vredendaal.

Some hard lattice meta-problems:

- Analyze cost of known attacks.
- Optimize attack parameters.
- Compare different attacks.
- Evaluate crypto parameters.
- Evaluate crypto designs.

`sntrup761` evaluations from "NTRU Prime: round 2" Table 2:

Ignoring cost of memory:

368	185	enum, ignoring hybrid
230	169	enum, including hybrid
153	139	sieving, ignoring hybrid
153	139	sieving, including hybrid

Accounting for cost of memory:

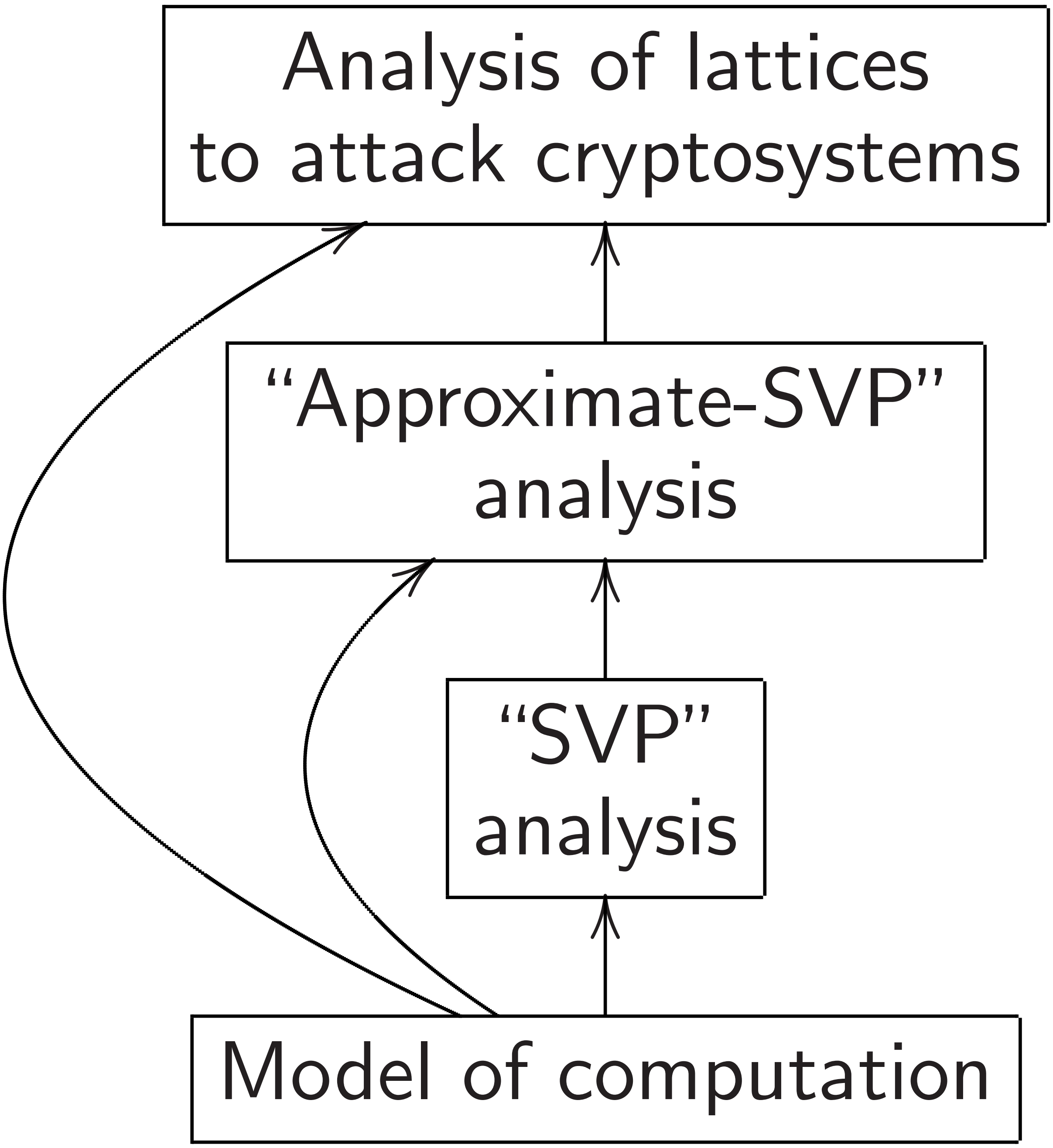
368	185	enum, ignoring hybrid
277	169	enum, including hybrid
208	208	sieving, ignoring hybrid
208	180	sieving, including hybrid

Security levels:

| . . . | pre-quantum

| . . . | post-quantum

Analysis of typical lattice attack has complications at four layers, and at interfaces between layers. This talk emphasizes top layer.



Three typical attack problems

Define $\mathcal{R} = \mathbf{Z}[x]/(x^{761} - x - 1)$;
"small" $=$ all coeffs in $\{-1, 0, 1\}$;
$w = 286$; $q = 4591$.

Attacker wants to find small weight-w secret $a \in \mathcal{R}$.

Problem 1: Public $G \in \mathcal{R}/q$ with $aG + e = 0$. Small secret $e \in \mathcal{R}$.

Problem 2: Public $G \in \mathcal{R}/q$ and $aG + e = A$. Small secret $e \in \mathcal{R}$.

Problem 3: Public $G_1, G_2 \in \mathcal{R}/q$.
Public $aG_1 + e_1, aG_2 + e_2$.
Small secrets $e_1, e_2 \in \mathcal{R}$.

Examples of target cryptosystems

Secret key: small a; small e.

Public key reveals multiplier G and approximation $A = aG + e$.

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Recognize similarity + credits:
"NTRU" $\Rightarrow$ Quotient NTRU.
"Ring-LWE" $\Rightarrow$ Product NTRU.

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Input small b, small d.

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Encryption for Product NTRU:

Input encoded message M.

Randomly generate

small b, small d, small c.

Ciphertext: $B = bG + d$

and $C = bA + M + c$.

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2019 Bernstein "Comparing proofs of security for lattice-based encryption" includes survey of G, a, e, c, M details and variants in NISTPQC submissions.

Lattices

Rewrite each problem as finding **short** nonzero solution to system of homogeneous $\mathcal{R}/q$ equations.

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Problem 3: Find $(a, t_1, t_2, e_1, e_2) \in \mathcal{R}^5$ with $aG_1 + e_1 = A_1 t_1$, $aG_2 + e_2 = A_2 t_2$, given $G_1, A_1, G_2, A_2 \in \mathcal{R}/q$.

Recognize each solution space as a full-rank lattice:

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Problem 3: Lattice is image of the map $(\overline{a}, \overline{t_1}, \overline{t_2}, \overline{r_1}, \overline{r_2}) \mapsto (\overline{a}, \overline{t_1}, \overline{t_2}, A_1\overline{t_1} + q\overline{r_1} - \overline{a}G_1, A_2\overline{t_2} + q\overline{r_2} - \overline{a}G_2)$.

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e.g. in Problem 2:
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Many more lattice vectors are fairly short combinations of independent vectors:
e.g., $((x+1)a, (x+1)t, (x+1)e)$.

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Other problems: same speedup. e.g. "Bai–Galbraith embedding" for Problem 2: Force $t \in \mathbf{Z}$; force a few coefficients of a to be 0.

(Slowdown if q is very large? Literature misses module option!)

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Lattice has rank $2 \cdot 761 = 1522$.
Attack parameter: $k = 13$.
Force k positions in a to be 0: restrict to sublattice of rank 1509.
Pr[a is in sublattice] $\approx 0.2\%$.

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Pretend this analysis applies to $\mathbf{Z}[x]/(x^{761} - x - 1)$. (It doesn't.)

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Attack parameter: $\lambda = 1.331876$. Rescaling (1997 Coppersmith–Shamir): Assign weight λ to positions in a. Increases length of a to $\lambda\sqrt{w} \approx 23$; increases det to $\lambda^{748} q^{600}$. (Is this λ optimal? Interaction with e size variation?)

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Hybrid attacks (2008 Howgrave-Graham, . . . , 2018 Wunderer): often faster; different analysis.