Lattice-based cryptography, part 1: simplicity

D. J. Bernstein

University of Illinois at Chicago;
Ruhr University Bochum

2000 Cohen cryptosystem

Public key: vector of integers
$K = (K_1, \ldots, K_N) \in \{-X, \ldots, X\}^N$.

Encryption:

1. Input message $m \in \{0, 1\}$.

2. Generate $r_1, \ldots, r_N \in \{0, 1\}$.
i.e. $r = (r_1, \ldots, r_N) \in \{0, 1\}^N$.

(Cohen says pick "half of the integers in the public key at random": I guess this means $N \in 2\mathbf{Z}$ and $\sum r_i = N/2$.)

3. Compute and send ciphertext
$C = (-1)^m(r_1K_1 + \cdots + r_NK_N)$.

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Key generation:
Generate $s \in \{1, .$
$u_1, \ldots, u_N \in \{0, .$
$K_i \in (u_i + s\mathbf{Z}) \cap$

Decryption:
$m = 0$ if C mod s
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Why this works:
K_i mod $s = u_i \leq$
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How can receiver decrypt?

Key generation:

Generate $s \in \{1, \ldots, Y\}$;

$u_1, \ldots, u_N \in \left\{0, \ldots, \left\lfloor \frac{s-1}{2N} \right.\right.$

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$m = 0$ if $C \bmod s \le (s-1)$

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Fedora: `dnf inst`

Source: `www.sage`

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Fedora: `dnf install sage`

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Web (use `print(X)` to see

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Sage is Python 3

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sage: 10^6 # power, not x
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```
sage: 10^6 # power, not xor
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sage: factor(31415926535897932 3)
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[14485, 7039, 6945, 15890
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```

For integers C, s with s > 0, Sage's "C%s" always produces outputs between 0 and $s - 1$.

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```

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sage: K=[ui+s*ra
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sage: sum(u)
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sage: u
[14485, 7039, 6945, 15890
 10493, 17333, 1397, 8656
 8213, 6370]
sage: sum(K)%s
96821
sage: sum(u)
96821
sage: s//2
179756
sage:
```

```
sage: K=[ui+s*randrange(
....:       ceil(-(X+ui)/s),
....:       floor((X-ui)/s)+1)
....:     for ui in u]
sage: K
[870056918917829,
 822006576592695,
 -294765544345815,
 -669275100080982,
 528958455221029,
 426006001074157,
 -641940176080531,
 501543495923784,
 -583064075392587,
 461093902438341]
```



```
sage: [Ki%s for Ki in K]
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
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```



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sage:
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sage: m=randrang
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sage: m
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sage: sum(r[i]*u
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sage: sum(K)%s
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sage: sum(u)
96821
sage: s//2
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sage:
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```
sage: m=randrange(2)
sage: r=[randrange(2)
....:      for i in range(N
sage: C=(-1)^m*sum(r[i]*K
....:     for i in range(N)
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Some problems with cryptos

1. Functionality problem:
System can't encrypt messa
that have more than 1 bit.

2. Security problem:
We want cryptosystems to r
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Decrypt $-C$. Flip result.

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sage: m=randrange(2)
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=randrange(2)
=[randrange(2)
  for i in range(N)]
=(-1)^m*sum(r[i]*K[i]
 for i in range(N))

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%s



um(r[i]*u[i]
    for i in range(N))
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B-bit input messa
$m = (m_1, \ldots, m_B$
For each $i \in \{1, \ldots$
Generate $r_{i,1}, \ldots,$

Ciphertext C:
$(-1)^{m_1}(r_{1,1}K_1 +$
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1. Transform 1-bit encryptio into multi-bit encryption by encrypting each bit separate
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B-bit input message
$m = (m_1, \ldots, m_B) \in \{0, 1\}^{B}$
For each $i \in \{1, \ldots, B\}$:
Generate $r_{i,1}, \ldots, r_{i,N} \in \{0,$

Ciphertext C:
$(-1)^{m_1}(r_{1,1}K_1 + \cdots + r_{1,N}K$
$\ldots,$
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2. Derandomize encryption,
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Derandomization: Generate
as cryptographic hash $H(m)$
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Decryption with reencryptio
1. Input C'. (Maybe $C' \neq C$
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Subset-s

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Subset-sum attack

Attacker searches
for $(r_1, \ldots, r_N)$,
checks $r_1K_1 + \cdots$
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This takes 2^N easy
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Modified attack:
For each $(r_1, \ldots, r$
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Subset-sum attacks

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Multi-target attack:
Apply this not just to B bits
one message, but all bits in
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Finding all bits in all messag
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Subset-sum attacks

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"This finds only one bit m_1."

— This is a problem in some applications. Should design encryption to leak *no* information.

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Modified attack:
For each $(r_1, \dots, r_N)$, look up $r_1 K_1 + \cdots + r_N K_N$ in hash table containing $\pm C_1, \pm C_2, \dots, \pm C_B$.

Multi-target attack:
Apply this not just to B bits in one message, but all bits in all messages sent to this key.

Finding all bits in all messages: total 2^N operations.

Finding 1% of all bits in all messages, huge information leak: total $0.01 \cdot 2^N$ operations.

sum attacks

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$\ldots, r_N)$,
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Apply this not just to B bits in one message, but all bits in all messages sent to this key.

Finding all bits in all messages: total 2^N operations.

Finding 1% of all bits in all messages, huge information leak: total $0.01 \cdot 2^N$ operations.

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sage: Y=
sage: X=
sage: s=
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sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*rand
sage: s
984887308997925
sage: u=[randran
....:    for i i
sage: u
[247, 418, 365,
 772, 209, 673,
sage:
```

ystem
e
with
K_N.

$\in 2\mathbf{Z}$

nds.

vi–
sZ;
N;

$\mathbf{Z}$.

Homomorphic encryption

If u_i/s is small enough then 2009 DGHV system is homomorphic.

Take two ciphertexts:
$C = m + 2\epsilon + sq$,
$C' = m' + 2\epsilon' + sq'$
with small $\epsilon, \epsilon' \in \mathbf{Z}$.

$C + C' = m + m' + 2(\epsilon + \epsilon') + s(q + q')$. This decrypts to $m + m'$ mod 2 if $\epsilon + \epsilon'$ is small.

$CC' = mm' + 2(\epsilon m' + \epsilon' m + 2\epsilon\epsilon') + s(\cdots)$. This decrypts to mm' if $\epsilon m' + \epsilon' m + 2\epsilon\epsilon'$ is small.

```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4
sage: s
984887308997925
sage: u=[randrange(E)
....:    for i in range(N
sage: u
[247, 418, 365, 738, 123,
 772, 209, 673, 47]
sage:
```

Homomorphic encryption

If u_i/s is small enough then 2009 DGHV system is homomorphic.

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$CC' = mm' + 2(\epsilon m' + \epsilon' m + 2\epsilon\epsilon') + s(\cdots)$. This decrypts to mm' if $\epsilon m' + \epsilon' m + 2\epsilon\epsilon'$ is small.


```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage: u=[randrange(E)
....:    for i in range(N)]
sage: u
[247, 418, 365, 738, 123, 735,
 772, 209, 673, 47]
sage:
```

orphic encryption

s small enough then 2009
ystem is homomorphic.

o ciphertexts:
$+ 2\epsilon + sq$,
$+ 2\epsilon' + sq'$
all $\epsilon, \epsilon' \in \mathbf{Z}$.

$= m + m' + 2(\epsilon + \epsilon') +$
). This decrypts to
mod 2 if $\epsilon + \epsilon'$ is small.

$mm' + 2(\epsilon m' + \epsilon' m + 2\epsilon\epsilon') +$
This decrypts to
$\epsilon m' + \epsilon' m + 2\epsilon\epsilon'$ is small.


```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage: u=[randrange(E)
....:    for i in range(N)]
sage: u
[247, 418, 365, 738, 123, 735,
 772, 209, 673, 47]
sage:
```

```
sage: K
....:
....:
....:
sage: K
[587473
 -11115
 794301
 688178
 742362
 102334
 -35716
 112142
 -11096
 -23562
```

ryption

ough then 2009
homomorphic.

xts:

q'
$\mathbf{Z}$.

$+ 2(\epsilon + \epsilon') +$
ecrypts to
$+ \epsilon'$ is small.

$n' + \epsilon' m + 2\epsilon\epsilon') +$
pts to
$+ 2\epsilon\epsilon'$ is small.


```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage: u=[randrange(E)
....:    for i in range(N)]
sage: u
[247, 418, 365, 738, 123, 735,
 772, 209, 673, 47]
sage:
```

```
sage: K=[2*ui+s*
....:       ceil(
....:       floor
....:    for ui
sage: K
[587473338058640
 -11115391791007
 79430145953378З
 688178021083749
 742362470968200
 102334582783153
 -35716867939855
 112142161911996
 -11096748622762
 -23562893778500
```

2009
hic.

) +

nall.

$2\epsilon\epsilon'$) +

small.


```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage: u=[randrange(E)
....:     for i in range(N)]
sage: u
[247, 418, 365, 738, 123, 735,
 772, 209, 673, 47]
sage:
```

```
sage: K=[2*ui+s*randrange
....:        ceil(-(X+2*ui)
....:        floor((X-2*ui)
....:     for ui in u]
sage: K
[587473338058640662659869
 -11115391791007200837703
 794301459533783434896055
 68817802108374958901751,
 742362470968200823035396
 102334582783153951505479
 -35716867939855887673000
 112142161911996460105144
 -11096748622762224955871
 -23562893778500377052338
```

```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage: u=[randrange(E)
....:    for i in range(N)]
sage: u
[247, 418, 365, 738, 123, 735,
 772, 209, 673, 47]
sage:
```



```
sage: K=[2*ui+s*randrange(
....:       ceil(-(X+2*ui)/s),
....:       floor((X-2*ui)/s)+1)
....:     for ui in u]
sage: K
[587473338058640662659869,
 -1111539179100720083770339,
 794301459533783434896055,
 688178021083749589017751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381]
```

```
=10
=2^10
=2^50
=2^80
=1+2*randrange(Y/4,Y/2)

08997925
=[randrange(E)
  for i in range(N)]

18, 365, 738, 123, 735,
09, 673, 47]
```



```
sage: K=[2*ui+s*randrange(
....:       ceil(-(X+2*ui)/s),
....:       floor((X-2*ui)/s)+1)
....:     for ui in u]
sage: K
[587473338058640662659869,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381]
```

```
sage: m
sage: r
....:
sage: C
....:
sage: C
2094088
sage: C
2703
sage: (
1
sage: m
1
sage:
```

```
range(Y/4,Y/2)

ge(E)
n range(N)]

738, 123, 735,
47]
```



```
sage: K=[2*ui+s*randrange(
....:       ceil(-(X+2*ui)/s),
....:       floor((X-2*ui)/s)+1)
....:     for ui in u]
sage: K
[587473338058640662659869,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381]
```



```
sage: m=randrang
sage: r=[randran
....:     for i i
sage: C=m+sum(r[
....:     for i i
sage: C
2094088748748247
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
,Y/2)
)]
 735,
```

```
sage: K=[2*ui+s*randrange(
....:      ceil(-(X+2*ui)/s),
....:      floor((X-2*ui)/s)+1)
....:    for ui in u]
sage: K
[587473338058640662659869,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381]
```



```
sage: m=randrange(2)
sage: r=[randrange(2)
....:    for i in range(N
sage: C=m+sum(r[i]*K[i]
....:    for i in range(N
sage: C
2094088748748247210016703
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
sage: K=[2*ui+s*randrange(
....:      ceil(-(X+2*ui)/s),
....:      floor((X-2*ui)/s)+1)
....:    for ui in u]
sage: K
[587473338058640662659869,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381]
```



```
sage: m=randrange(2)
sage: r=[randrange(2)
....:    for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:    for i in range(N))
sage: C
2094088748748247210016703
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
=[2*ui+s*randrange(
    ceil(-(X+2*ui)/s),
    floor((X-2*ui)/s)+1)
  for ui in u]

33805864066265986 9,
39179100720083770339,
45953378343489605 5,
02108374958901751,
47096820082303539 6,
58278315395150547 95,
86793985588767300 06,
16191199646010514 43,
74862276222495587129,
89377850037052338 1]
```



```
sage: m=randrange(2)
sage: r=[randrange(2)
....:    for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:    for i in range(N))
sage: C
2094088748748247210016703
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
sage: m
sage: r
....:
sage: C
....:
sage: C
-517223
sage: C
4971
sage: (
1
sage: m
1
sage:
```

```
randrange(
-(X+2*ui)/s),
((X-2*ui)/s)+1)
in u]

662659869,
20083770339,
434896055,
58901751,
823035396,
9515054795,
8876730006,
4601051443,
22495587129,
3770523381]
```



```
sage: m=randrange(2)
sage: r=[randrange(2)
....:     for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:     for i in range(N))
sage: C
20940887487482472100167032
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
sage: m2=randran
sage: r2=[randra
....:     for i
sage: C2=m2+sum(
....:     for i
sage: C2
-517223537379827
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```

```
(
/s),
/s)+1)

,
39,
,

,
5,
6,
3,
29,
1]
```



```
sage: m=randrange(2)
sage: r=[randrange(2)
....:     for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:     for i in range(N))
sage: C
20940887487482472100167O3
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:     for i in range(
sage: C2=m2+sum(r2[i]*K[i
....:     for i in range(
sage: C2
-5172235373798273727012 9
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```

```
sage: m=randrange(2)
sage: r=[randrange(2)
....:     for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:     for i in range(N))
sage: C
20940887487482472100016703
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```



```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:     for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:     for i in range(N))
sage: C2
-51722353737982737270129
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```

```
=randrange(2)
=[randrange(2)
 for i in range(N)]
=m+sum(r[i]*K[i]
 for i in range(N))

74874824721001670з
%s

C%s)%2
```



```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:     for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:     for i in range(N))
sage: C2
-51722353737982737270129
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```

```
sage: (C
7674
sage: (C
13436613
sage:
```

Because
are smal
have C –
(C' mod
(C mod

Refineme
to ciphe
Gentry)

```
e(2)
ge(2)
n range(N)]
i]*K[i]
n range(N))

210016703
```



```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:     for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:     for i in range(N))
sage: C2
-51722353737982737270129
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```

```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

Because C mod s
are small enough
have $C + C'$ mod s
(C' mod s) and C
(C mod s)(C' mod

Refinements: add
to ciphertexts, boo
Gentry) to control

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:     for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:     for i in range(N))
sage: C2
-51722353737982737270129
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```



```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

Because C mod s and C' mo
are small enough compared
have $C + C'$ mod $s = (C$ mo
$(C'$ mod $s)$ and CC' mod s
$(C$ mod $s)(C'$ mod $s)$.

Refinements: add more nois
to ciphertexts, bootstrap (20
Gentry) to control noise, etc

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:     for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:     for i in range(N))
sage: C2
-5172235373798273727O129
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
sage:
```



```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

Because C mod s and C' mod s are small enough compared to s, have $C + C'$ mod $s = (C$ mod $s) +$ $(C'$ mod $s)$ and CC' mod $s =$ $(C$ mod $s)(C'$ mod $s)$.

Refinements: add more noise to ciphertexts, bootstrap (2009 Gentry) to control noise, etc.

```
2=randrange(2)
2=[randrange(2)
   for i in range(N)]
2=m2+sum(r2[i]*K[i]
   for i in range(N))
2
53737982737270129
2%s

C2%s)%2

2
```



```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

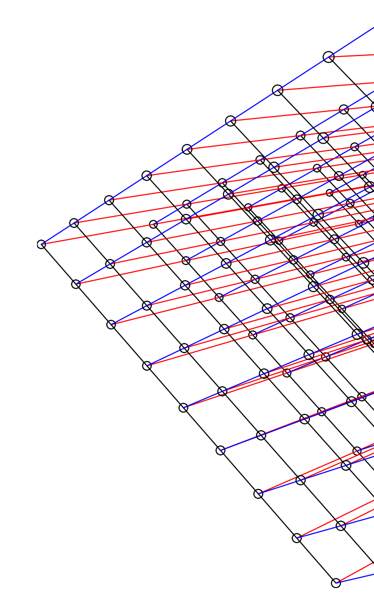
Because C mod s and C' mod s are small enough compared to s, have $C + C'$ mod $s = (C$ mod $s) + (C'$ mod $s)$ and CC' mod s $=$ $(C$ mod $s)(C'$ mod $s)$.

Refinements: add more noise to ciphertexts, bootstrap (2009 Gentry) to control noise, etc.

Lattices

This is a

This is a



```
ge(2)
nge(2)
in range(N)]
r2[i]*K[i]
in range(N))

37270129
```



```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

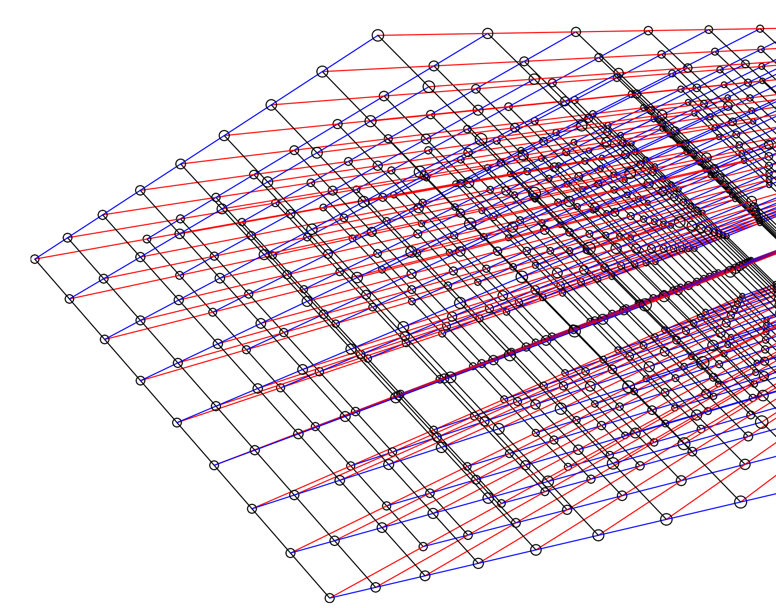
Because C mod s and C' mod s are small enough compared to s, have $C + C'$ mod $s = (C$ mod $s) + (C'$ mod $s)$ and CC' mod $s = (C$ mod $s)(C'$ mod $s)$.

Refinements: add more noise to ciphertexts, bootstrap (2009 Gentry) to control noise, etc.

Lattices

This is a lettuce:

This is a lattice:



```
N)]
]
N))
```



```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

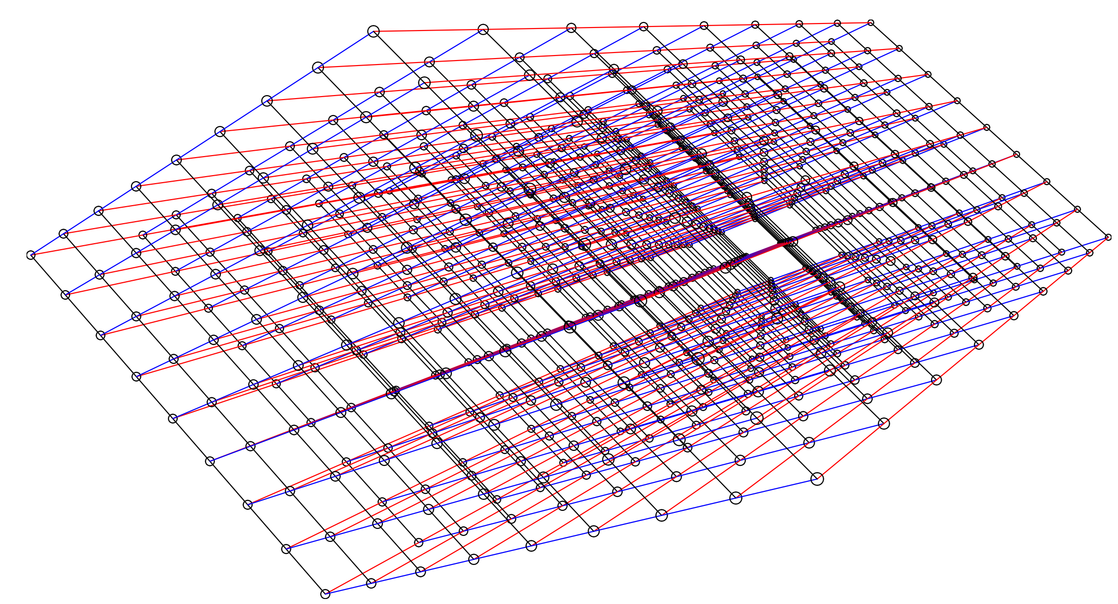
Because C mod s and C' mod s are small enough compared to s, have $C + C'$ mod $s = (C$ mod $s) + (C'$ mod $s)$ and CC' mod $s = (C$ mod $s)(C'$ mod $s)$.

Refinements: add more noise to ciphertexts, bootstrap (2009 Gentry) to control noise, etc.

Lattices

This is a lettuce:

This is a lattice:



```
sage: (C+C2)%s
7674
sage: (C*C2)%s
13436613
sage:
```

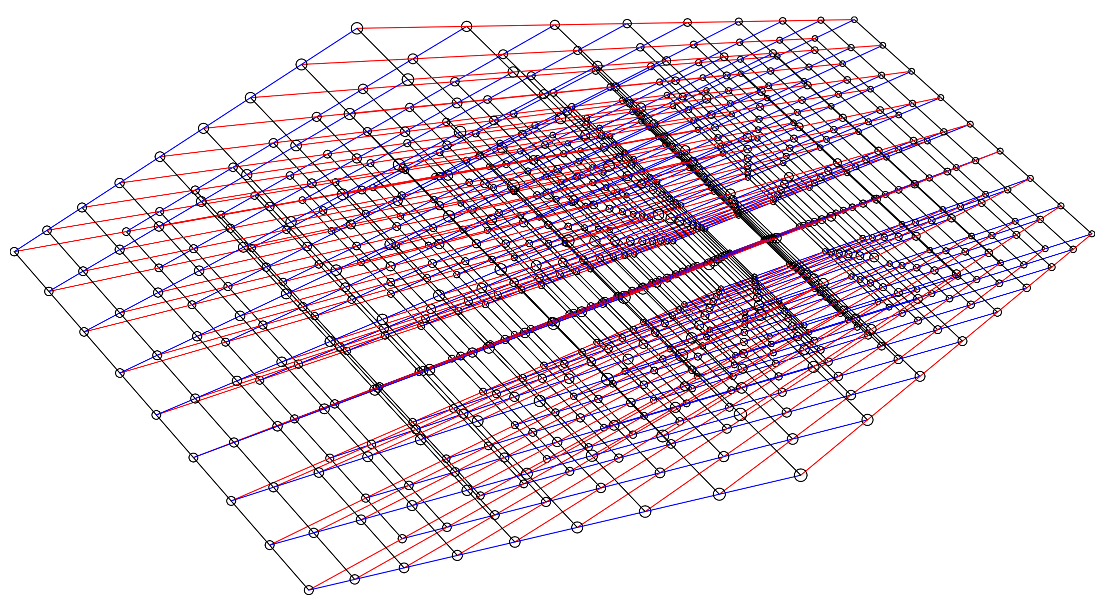
Because C mod s and C' mod s are small enough compared to s, have $C + C'$ mod $s = (C$ mod $s) + (C'$ mod $s)$ and CC' mod $s = (C$ mod $s)(C'$ mod $s)$.

Refinements: add more noise to ciphertexts, bootstrap (2009 Gentry) to control noise, etc.

Lattices

This is a lettuce:

This is a lattice:



```
C+C2)%s
```

```
C*C2)%s
3
```

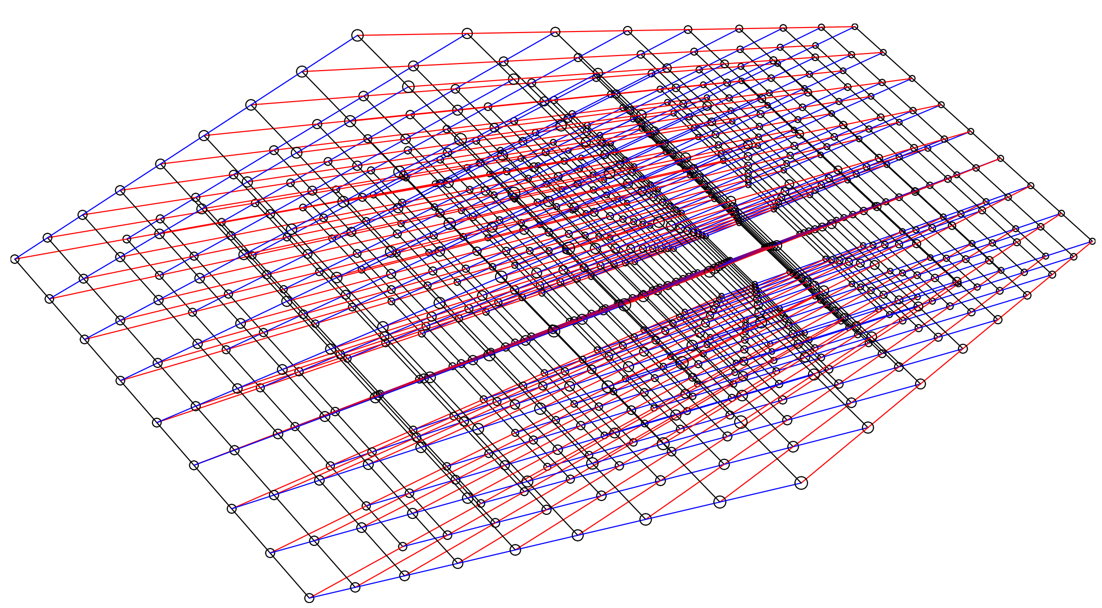
C mod s and C' mod s
l enough compared to s,
$+\,C'$ mod $s = (C$ mod $s) +$
$s)$ and CC' mod s $=$
$s)(C'$ mod $s)$.

ents: add more noise
rtexts, bootstrap (2009
to control noise, etc.

Lattices

This is a lettuce:

This is a lattice:



Lattices,

Assume
are **R**-lin
i.e., $\mathbf{R}V_1$
$\{r_1V_1 +$
is a D-d

$\mathbf{Z}V_1 + \cdots$
$\{r_1V_1 +$
is a rank

$V_1, \ldots, V$
is a **basi**

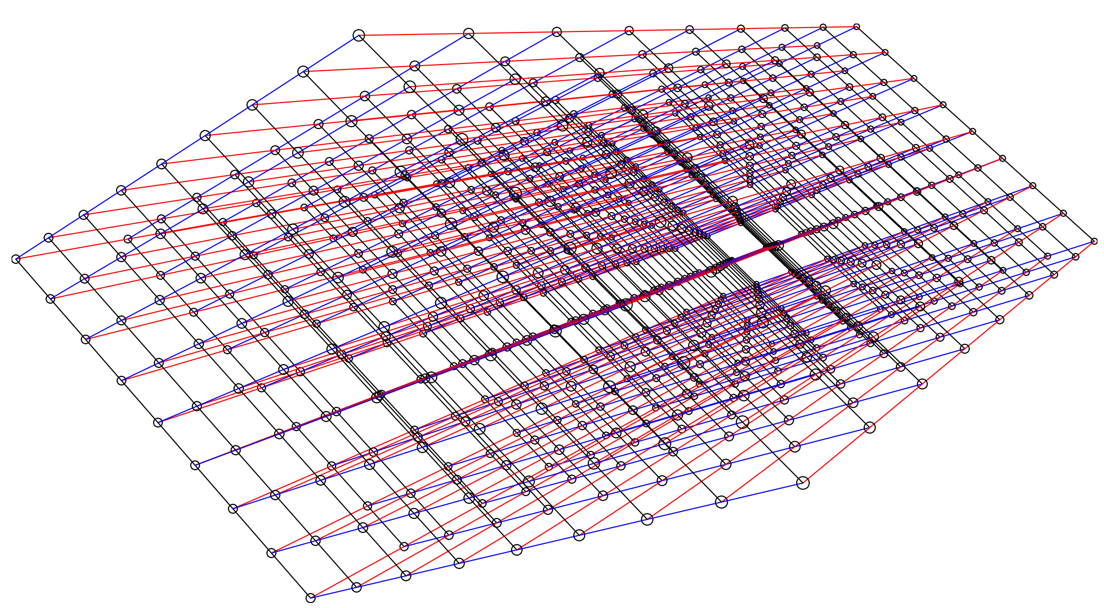
and C' mod s
compared to s,
$s = (C \bmod s) +$
$C' \bmod s =$
s).

more noise
otstrap (2009
noise, etc.

Lattices

This is a lettuce:

This is a lattice:



Lattices, mathema

Assume that $V_1, \ldots$
are **R**-linearly inde
i.e., $\mathbf{R}V_1 + \cdots + \mathbf{R}$
$\{r_1V_1 + \cdots + r_DV$
is a D-dimensiona

$\mathbf{Z}V_1 + \cdots + \mathbf{Z}V_D =$
$\{r_1V_1 + \cdots + r_DV$
is a rank-D length

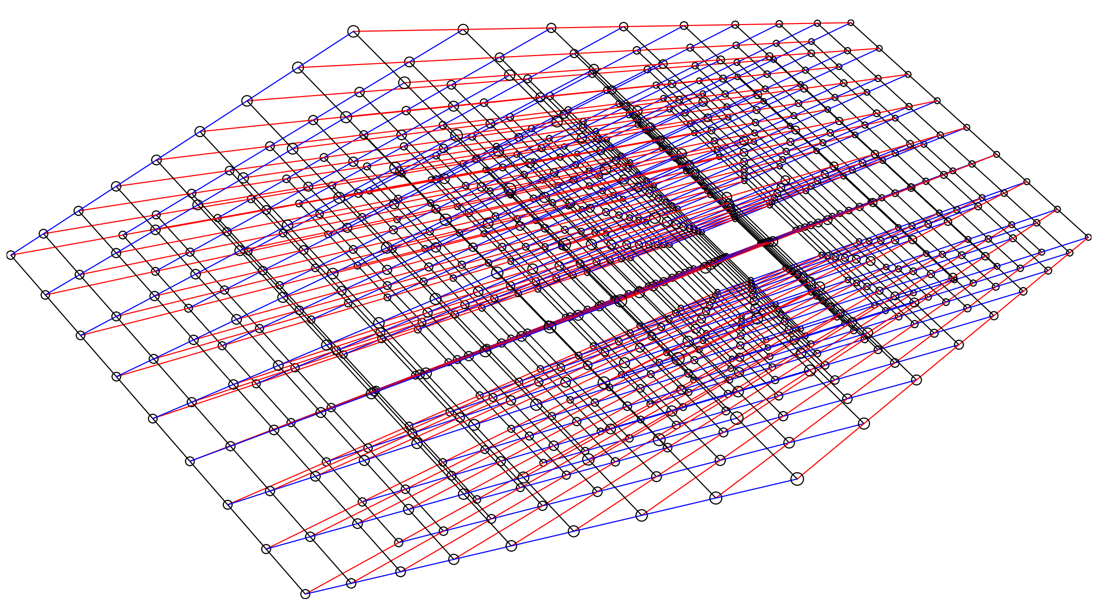
$V_1, \ldots, V_D$
is a **basis** of this l

od s
to s,
od $s) +$
$=$
e
009
c.

Lattices

This is a lettuce:

This is a lattice:



Lattices, mathematically

Assume that $V_1, \ldots, V_D \in \mathbf{R}$
are $\mathbf{R}$-linearly independent,
i.e., $\mathbf{R}V_1 + \cdots + \mathbf{R}V_D =$
$\{r_1V_1 + \cdots + r_DV_D : r_1, \ldots,$
is a D-dimensional vector sp

$\mathbf{Z}V_1 + \cdots + \mathbf{Z}V_D =$
$\{r_1V_1 + \cdots + r_DV_D : r_1, \ldots,$
is a rank-D length-N **lattice**

$V_1, \ldots, V_D$
is a **basis** of this lattice.

Lattices

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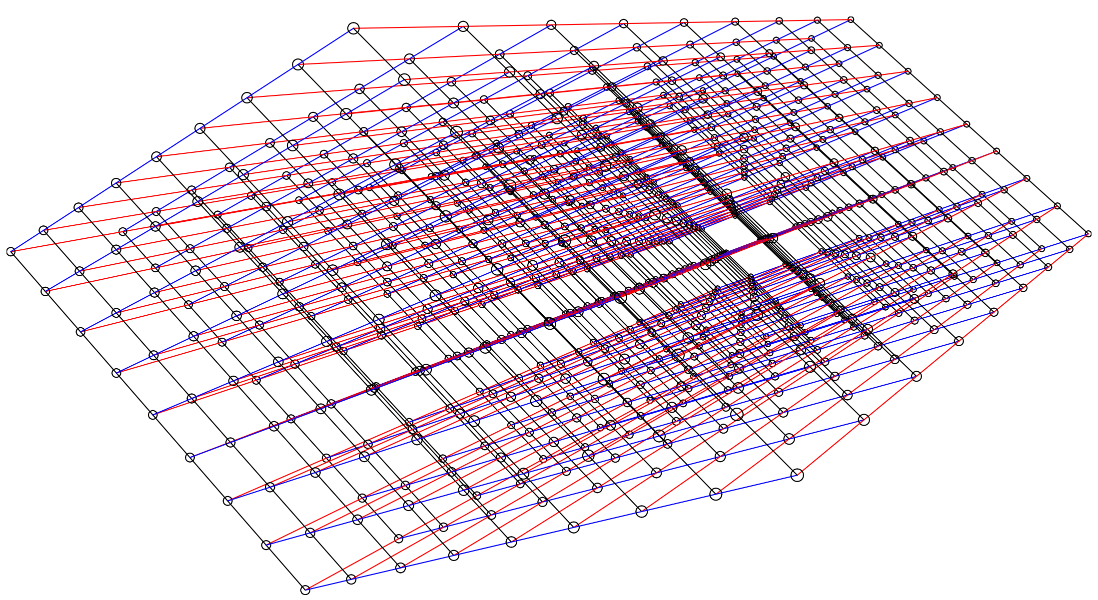

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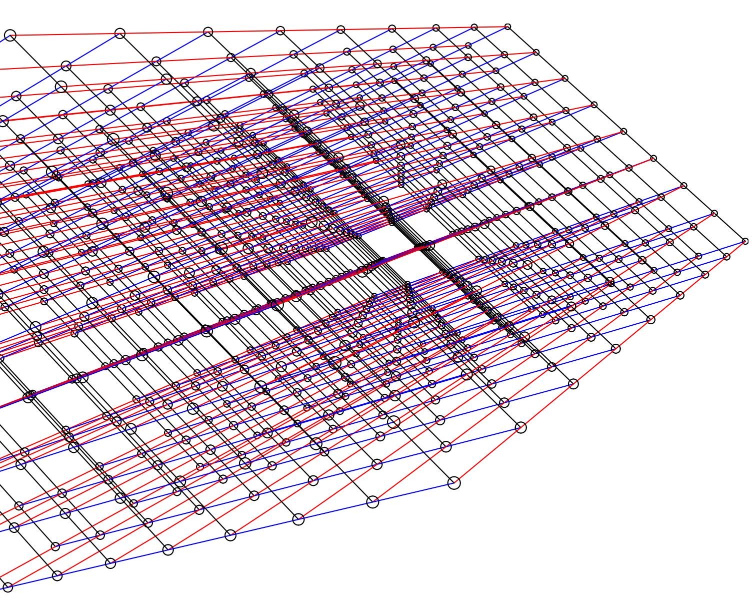

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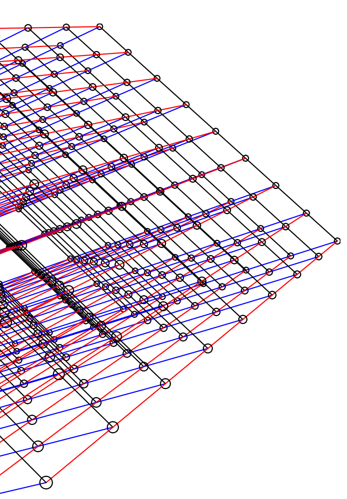

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sage: V=matrix.i
sage: V=-K[0]*V
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```
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Lattice attacks on DGHV keys

Recall $K_i = 2u_i + sq_i \approx sq_i$.
Each u_i is small: $u_i < E$.
Note $q_j K_i - q_i K_j = 2q_j u_i - 2q_i u_j$.

Define
$V_1 = (E, K_2, K_3, \ldots, K_N)$;
$V_2 = (0, -K_1, 0, \ldots, 0)$;
$V_3 = (0, 0, -K_1, \ldots, 0)$;
$\ldots$;
$V_N = (0, 0, 0, \ldots, -K_1)$.

Define $L = \mathbf{Z}V_1 + \cdots + \mathbf{Z}V_N$.
L contains $q_1 V_1 + \cdots + q_N V_N =$
$(q_1 E, q_1 K_2 - q_2 K_1, \ldots) =$
$(q_1 E, 2q_1 u_2 - 2q_2 u_1, \ldots)$.


```
sage: V=matrix.identity(N)
sage: V=-K[0]*V
sage: Vtop=copy(K)
sage: Vtop[0]=E
sage: V[0]=Vtop
sage: q0=V.LLL()[0][0]/E
sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage: s
984887308997925
sage:
```

attacks on DGHV keys

$K_i = 2u_i + sq_i \approx sq_i$.

is small: $u_i < E$.

$K_i - q_iK_j = 2q_ju_i - 2q_iu_j$.

$, K_2, K_3, \ldots, K_N)$;

$-K_1, 0, \ldots, 0)$;

$0, -K_1, \ldots, 0)$;

$, 0, 0, \ldots, -K_1)$.

$= \mathbf{Z}V_1 + \cdots + \mathbf{Z}V_N$.

ns $q_1V_1 + \cdots + q_NV_N =$

$K_2 - q_2K_1, \ldots) =$

$_1u_2 - 2q_2u_1, \ldots)$.


```
sage: V=matrix.identity(N)
sage: V=-K[0]*V
sage: Vtop=copy(K)
sage: Vtop[0]=E
sage: V[0]=Vtop
sage: q0=V.LLL()[0][0]/E
sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage: s
984887308997925
sage:
```

```
sage: V
(1024,
 -11115
 794301
 688178
 742362
 102334
 -35716
 112142
 -11096
 -23562
sage: V
(0, -58
 0, 0, 
sage:
```

DGHV keys

$sq_i \approx sq_i$.

$u_i < E$.

$= 2q_j u_i - 2q_i u_j$.

$\ldots, K_N)$;

$\ldots, 0)$;

$\ldots, 0)$;

$-K_1)$.

$\cdots + \mathbf{Z}V_N$.

$\cdots + q_N V_N =$

$_1, \ldots) =$

$u_1, \ldots)$.


```
sage: V=matrix.identity(N)
sage: V=-K[0]*V
sage: Vtop=copy(K)
sage: Vtop[0]=E
sage: V[0]=Vtop
sage: q0=V.LLL()[0][0]/E
sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage: s
984887308997925
sage:
```

```
sage: V[0]
(1024,
 -11115391791007
 79430145953378З
 68817802108374
 74236247096820
 10233458278315
 -3571686793985
 11214216191199
 -1109674862276
 -2356289377850
sage: V[1]
(0, -5874733380
 0, 0, 0, 0, 0,
sage:
```

eys

.

$2q_i u_j$.

N.
$/_N =$

```
sage: V=matrix.identity(N)
sage: V=-K[0]*V
sage: Vtop=copy(K)
sage: Vtop[0]=E
sage: V[0]=Vtop
sage: q0=V.LLL()[0][0]/E
sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage: s
984887308997925
sage:
```



```
sage: V[0]
(1024,
 -1111539179100720083770З
 79430145953378343489605‍5
 68817802108374958901751,
 742362470968200823035396
 102334582783153951505479
 -357168679398558876730000
 112142161911996460105144
 -110967486227622249558871
 -235628937785003770523338
sage: V[1]
(0, -587473338058640662 65
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```

```
sage: V=matrix.identity(N)
sage: V=-K[0]*V
sage: Vtop=copy(K)
sage: Vtop[0]=E
sage: V[0]=Vtop
sage: q0=V.LLL()[0][0]/E
sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage: s
984887308997925
sage:
```



```
sage: V[0]
(1024,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381)
sage: V[1]
(0, -587473338058640662659869,
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```

```
=matrix.identity(N)
=-K[0]*V
top=copy(K)
top[0]=E
[0]=Vtop
0=V.LLL()[0][0]/E
0
75
ound(K[0]/q0)
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```



```
sage: V[0]
(1024,
 -11115391791007200837703339,
 7943014595337834348960 55,
 6881780210837495890175 1,
 7423624709682008230353 96,
 10233458278315395150547 95,
 -35716867939855887673000 6,
 11214216191199646010514 43,
 -11096748622762224955871 29,
 -23562893778500377052338 1)
sage: V[1]
(0, -58747333805864066265986 9,
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```

```
sage: V
(610803
 370302
 -22561
 110012
 135946
sage: q
sage: q
6108035
sage: q
1056189
sage: q
1742566
sage:
```

```
dentity(N)

K)

[0][0]/E

/q0)
```



```
sage: V[0]
(1024,
 -1115391791007200837703339,
 794301459533783434896055,
 688178021083749589017751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381)
sage: V[1]
(0, -587473338058640662659869,
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```

```
sage: V.LLL()[0]
(610803584000, 1
 37030242384, 84
 -225618319442,
 1100126026284,
 1359463649048,
sage: q=[Ki//s f
sage: q[0]*E
610803584000
sage: q[0]*K[1]-
1056189937254
sage: q[0]*K[9]-
174256676348
sage:
```

])


```
sage: V[0]
(1024,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381)
sage: V[1]
(0, -587473338058640662659869,
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```



```
sage: V.LLL()[0]
(610803584000, 1056189937
 37030242384, 84589845469
 -225618319442, 363547143
 1100126026284, -31315097
 1359463649048, 174256676
sage: q=[Ki//s for Ki in
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
174256676348
sage:
```

```
sage: V[0]
(1024,
 -1111539179100720083770339,
 794301459533783434896055,
 688178021083749589017751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -2356289377850037705233881)
sage: V[1]
(0, -5874733380586406626598869,
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```



```
sage: V.LLL()[0]
(610803584000, 1056189937254,
 37030242384, 845898454698,
 -225618319442, 363547143644,
 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
174256676348
sage:
```

```
[0]

39179100720083770339,
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470968200823035396,
5827831539515054795,
8679398558876730006,
1619119964601051443,
74862276222495587129,
893778500370523381)
[1]
7473338058640662659869,
0, 0, 0, 0, 0, 0)
```



```
sage: V.LLL()[0]
(610803584000, 1056189937254,
 37030242384, 845898454698,
 -225618319442, 363547143644,
 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
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0, 0, 0)
```



```
sage: V.LLL()[0]
(610803584000, 1056189937254,
 37030242384, 845898454698,
 -225618319442, 363547143644,
 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
174256676348
sage:
```

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339,
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sage: V.LLL()[0]
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 -225618319442, 363547143644,
 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
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sage:
```


2009 DGHV analysis:
can choose key sizes where
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2011 Coron–Mandal–Naccac
Tibouchi: reduce key sizes
by modifying DGHV. "This
shows that fully homomorph
encryption can be implemen
with a simple scheme."

e.g. all attacks take $\geq 2^{72}$ cy
with public keys only 802ME

2012 Chen–Nguyen: faster a
Need bigger DGHV/CMNT

```
sage: V.LLL()[0]
(610803584000, 1056189937254,
 37030242384, 845898454698,
 -225618319442, 363547143644,
 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
174256676348
sage:
```


2009 DGHV analysis: can choose key sizes where these lattice attacks fail.

2011 Coron–Mandal–Naccache–Tibouchi: reduce key sizes by modifying DGHV. "This shows that fully homomorphic encryption can be implemented with a simple scheme."

e.g. all attacks take $\geq 2^{72}$ cycles with public keys only 802MB.

2012 Chen–Nguyen: faster attack. Need bigger DGHV/CMNT keys.

```
.LLL()[0]
584000, 1056189937254,
42384, 845898454698,
3319442, 363547143644,
6026284, -313150978512,
3649048, 174256676348)
=[Ki//s for Ki in K]
[0]*E
84000
[0]*K[1]-q[1]*K[0]
937254
[0]*K[9]-q[9]*K[0]
76348
```


2009 DGHV analysis: can choose key sizes where these lattice attacks fail.

2011 Coron–Mandal–Naccache–Tibouchi: reduce key sizes by modifying DGHV. "This shows that fully homomorphic encryption can be implemented with a simple scheme."

e.g. all attacks take $\geq 2^{72}$ cycles with public keys only 802MB.

2012 Chen–Nguyen: faster attack. Need bigger DGHV/CMNT keys.

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or Ki in K]

q[1]*K[0]

q[9]*K[0]
```


2009 DGHV analysis: can choose key sizes where these lattice attacks fail.

2011 Coron–Mandal–Naccache–Tibouchi: reduce key sizes by modifying DGHV. "This shows that fully homomorphic encryption can be implemented with a simple scheme."

e.g. all attacks take $\geq 2^{72}$ cycles with public keys only 802MB.

2012 Chen–Nguyen: faster attack. Need bigger DGHV/CMNT keys.

Big attack surface

1991 Chaum–van

Pfitzmann: choose

define $C(x, y) = 4$

for suitable ranges

Simple, beautiful,

Very easy security

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2009 DGHV analysis: can choose key sizes where these lattice attacks fail.

2011 Coron–Mandal–Naccache–Tibouchi: reduce key sizes by modifying DGHV. "This shows that fully homomorphic encryption can be implemented with a simple scheme."

e.g. all attacks take $\geq 2^{72}$ cycles with public keys only 802MB.

2012 Chen–Nguyen: faster attack. Need bigger DGHV/CMNT keys.

Big attack surfaces are dang

1991 Chaum–van Heijst–Pfitzmann: choose p sensibl define $C(x, y) = 4^x 9^y \bmod$ for suitable ranges of x and

Simple, beautiful, structured Very easy security reduction finding C collision implies computing a discrete logarit

Typical exaggerations: C is "provably secure"; C is "cryptographically collision-f "security follows from rigoro mathematical proofs".

2009 DGHV analysis:
can choose key sizes where
these lattice attacks fail.

2011 Coron–Mandal–Naccache–Tibouchi: reduce key sizes by modifying DGHV. "This shows that fully homomorphic encryption can be implemented with a simple scheme."

e.g. all attacks take $\geq 2^{72}$ cycles with public keys only 802MB.

2012 Chen–Nguyen: faster attack. Need bigger DGHV/CMNT keys.

Big attack surfaces are dangerous

1991 Chaum–van Heijst–Pfitzmann: choose p sensibly; define $C(x, y) = 4^x 9^y \bmod p$ for suitable ranges of x and y.

Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

Typical exaggerations:
C is "provably secure"; C is "cryptographically collision-free"; "security follows from rigorous mathematical proofs".

GHV analysis:
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Big attack surfaces are dangerous

1991 Chaum–van Heijst–Pfitzmann: choose p sensibly; define $C(x, y) = 4^x 9^y \bmod p$ for suitable ranges of x and y.

Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

Typical exaggerations:
C is "provably secure"; C is "cryptographically collision-free"; "security follows from rigorous mathematical proofs".

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Big attack surfaces are dangerous

1991 Chaum–van Heijst–Pfitzmann: choose p sensibly; define $C(x, y) = 4^x 9^y \bmod p$ for suitable ranges of x and y.

Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

Typical exaggerations:
C is “provably secure”; C is “cryptographically collision-free”; “security follows from rigorous mathematical proofs”.

Security losses in C
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1991 Chaum–van Heijst–Pfitzmann: choose p sensibly; define $C(x, y) = 4^x 9^y \bmod p$ for suitable ranges of x and y.

Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

Typical exaggerations: C is "provably secure"; C is "cryptographically collision-free"; "security follows from rigorous mathematical proofs".

Security losses in C include
1922 Kraitchik (index calcul
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C is very bad cryptography.
No matter what user's cost
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Big attack surfaces are dangerous

1991 Chaum–van Heijst–Pfitzmann: choose p sensibly; define $C(x, y) = 4^x 9^y \bmod p$ for suitable ranges of x and y.

Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

Typical exaggerations: C is "provably secure"; C is "cryptographically collision-free"; "security follows from rigorous mathematical proofs".

Security losses in C include
1922 Kraitchik (index calculus);
1986 Coppersmith–Odlyzko–Schroeppel (NFS predecessor);
1993 Gordon (general DL NFS);
1993 Schirokauer (faster NFS);
1994 Shor (quantum poly time);
many subsequent attack speedups from people who care about pre-quantum security.

C is very bad cryptography. No matter what user's cost limit is, obtain better security with "unstructured" compression-function designs such as BLAKE.

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Security losses in C include
1922 Kraitchik (index calculus);
1986 Coppersmith–Odlyzko–
Schroeppel (NFS predecessor);
1993 Gordon (general DL NFS);
1993 Schirokauer (faster NFS);
1994 Shor (quantum poly time);
many subsequent attack speedups
from people who care about
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C is very bad cryptography.
No matter what user's cost limit
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Security losses in C include
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1986 Coppersmith–Odlyzko–Schroeppel (NFS predecessor);
1993 Gordon (general DL NFS);
1993 Schirokauer (faster NFS);
1994 Shor (quantum poly time);
many subsequent attack speedups from people who care about pre-quantum security.

C is very bad cryptography. No matter what user's cost limit is, obtain better security with "unstructured" compression-function designs such as BLAKE.

For public-key enc
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Security losses in C include
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1986 Coppersmith–Odlyzko–Schroeppel (NFS predecessor);
1993 Gordon (general DL NFS);
1993 Schirokauer (faster NFS);
1994 Shor (quantum poly time);
many subsequent attack speedups from people who care about pre-quantum security.

C is very bad cryptography. No matter what user's cost limit is, obtain better security with "unstructured" compression-function designs such as BLAKE.

For public-key encryption:
Some mathematical structur
seems to be unavoidable,
but pursuing simple structur
often leads to security disast

Pre-quantum example: DH
simpler than ECDH, but DH
suffered many more security
than ECDH. State-of-the-ar
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2013 Barbulescu–Gaudry–Jo
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Security losses in C include
1922 Kraitchik (index calculus);
1986 Coppersmith–Odlyzko–Schroeppel (NFS predecessor);
1993 Gordon (general DL NFS);
1993 Schirokauer (faster NFS);
1994 Shor (quantum poly time);
many subsequent attack speedups from people who care about pre-quantum security.

C is very bad cryptography. No matter what user's cost limit is, obtain better security with "unstructured" compression-function designs such as BLAKE.

For public-key encryption: Some mathematical structure seems to be unavoidable, but pursuing simple structures often leads to security disasters.

Pre-quantum example: DH is simpler than ECDH, but DH has suffered many more security losses than ECDH. State-of-the-art DH attacks are very complicated.

2013 Barbulescu–Gaudry–Joux–Thomé: pre-quantum quasi-poly break of small-characteristic DH.

losses in C include
aitchik (index calculus);
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For public-key encryption:
Some mathematical structure
seems to be unavoidable,
but pursuing simple structures
often leads to security disasters.

Pre-quantum example: DH is
simpler than ECDH, but DH has
suffered many more security losses
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For public-key encryption:
Some mathematical structure seems to be unavoidable, but pursuing simple structures often leads to security disasters.

Pre-quantum example: DH is simpler than ECDH, but DH has suffered many more security losses than ECDH. State-of-the-art DH attacks are very complicated.

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The state-of-the-art attacks against Cohen's cryptosystem are much more complicated than the cryptosystem is. Scary!

Lattice-based cryptosystems are advertised as "algorithmically simple", consisting mainly of "linear operations on vectors". Attacks exploit this structure!

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